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ON THE GEOMETRY OF LINEAR PASSIVE SYSTEMS

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ABSTRACT. Passive systems are of interest in many areas of system theory, including network synthesis, stochastic realization and optimal control. Linear passive systems are externally characterized by positive real transfer functions. This paper is concerned with the geometry of rational positive real functions.

Extending the work of R. W. Brockett, we characterize here the homotopic equivalence classes of rational positive real functions. We also obtain parametrizations of these equivalence classes in special cases. One of our main results leads to a geometric interpretation of Darlington synthesis. Our approach rests on certain interesting group theoretic facts about passive systems.

The results of this paper should be viewed as part of a general program to understand the geometry of families of systems and related system-theoretic questions. In this connection we direct attention to the papers [1], [2], [3].

1. INTRODUCTION. Consider a finite-dimensional linear dynamic system (with constant coefficients):

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} , \qquad (*)$$

The system (*) is said to be *passive* (dissipative) if there exists a positive definite matrix K such that along solutions of (*) the following dissipation inequality [5] holds:

$$\frac{d}{dt} \langle x(t), Kx(t) \rangle \le \langle u(t), y(t) \rangle. \qquad (**)$$

It is well-known [6] that passivity has a frequency-domain characterization: if the system (*) is minimal, then it is passive iff the transfer function, $G(s) = D + C(sI - A)^{-1}B$ is a *positive real function*, i.e., $G(s)$ is analytic and $G(\sigma + j\omega) + G'(\sigma - j\omega)$ is non-negative definite for $\sigma > 0$. Equivalently,

(i) $\mathrm{Re}\,\lambda(A) \le 0$ for $\lambda(A) \in$ spectrum (A),

(ii) $G(j\omega) + G'(-j\omega) \ge 0$ for all real ω, $j\omega \notin$ spectrum (A),

(iii) The eigenvalues of A with $\mathrm{Re}\,\lambda(A) = 0$ are non-repeated and the residue matrix at those eigenvalues is Hermitian and non-negative definite.

Anderson [7] has several useful results characterizing rational positive real matrix valued functions via certain algebraic conditions on the realizations of such functions. It is further well-known that a rational positive real scalar function can be realized as the impedance/admittance of an electrical network composed of resistors, capacitors, inductors, transformers and gyrators [8]. An interesting subclass of (*) consists of *lossless systems* that satisfy the dissipation inequality (**) with equality. In the context of the stochastic realization problem passive systems appear as follows. Consider the Stochastic differential system driven by Gaussian white noise:

$$\begin{aligned} dx &= Ax\,dt + b\,dw \\ y &= cx\,. \end{aligned} \qquad (1.1)$$

Suppose the triple $[A,b,c]$ is minimal and $\mathrm{Re}[\lambda(A)] < 0$. Then (1.1) admits a smooth nondegenerate invariant Gaussian measure and if we denote the corresponding correlation function,

$$r_{yy}(\tau) = E[y_{t+\tau} y_t]\,, \qquad (1.2)$$

then, the Laplace transform,

$$Z(s) = \int_0^\infty r_{yy}(\tau)\, e^{-s\tau} d\tau \qquad (1.3)$$

is a rational positive real function (a consequence of Bochner's theorem). The problem of spectral factorization is to pass from $r_{yy}(\cdot)$ to the transfer function $g(s) = c(sI - A)^{-1}b$ of equation (1.1) by solving the equations

$$\phi(s) = g(s)g(-s) = Z(s) + Z(-s) , \tag{1.4}$$

where, $\phi(j\omega)$ is the Fourier transform of $r_{yy}(\tau)$. This also forms the basis of some identification algorithms (c.f. Mehra [9] for discrete-time version). The underlying "parameter space" in these problems is the set of positive real functions. Our goal here is to investigate the global properties of this set. Before we proceed along these lines, we set down in the remainder of this section some preliminaries.

In [10], Roger Brockett initiated a program for the study of the space Rat(n) of rational functions. The analytic manifold Rat(n) is defined as follows: Consider the set of rational functions of the form $g(s) = q(s)/p(s)$ where $q(s) = q_{n-1}s^{n-1} + \ldots + q_s$ and $p(s) = s^n + p_{n-1}s^{n-1} + \ldots + p_o$ are *relatively prime* polynomials, as an open subspace of $\mathbb{R}^{2n}$. This subspace together with the manifold structure from $\mathbb{R}^{2n}$ is called Rat(n). Now, given any rational function $g(s) \in$ Rat(n) we can define the Cauchy index as a winding number

$$I_{-\infty}^{\infty}(g) = (\text{number of jumps of } g(s) \text{ from } -\infty \text{ to } +\infty) - (\text{number of jumps of } g(s) \text{ from } +\infty \text{ to } -\infty)$$

as s ranges over the reals from $-\infty$ to $+\infty$. The Cauchy index is tied up in a fundamental way with the topology of rational functions. In [10] Brockett showed that:

(a) Rat(n) splits; $\text{Rat}(n) = \bigcup_{\mu+\nu=n} \text{Rat}(\mu,\nu)$ where on each component $\text{Rat}(\mu,\nu)$ the Cauchy index is constant and takes the value $(\mu-\nu)$.

(b) $\text{Rat}(n,o) \simeq \text{Rat}(o,n) \simeq \mathbb{R}^{2n}$

(c) $\text{Rat}(1,n-1) \simeq \text{Rat}(n-1,1) \simeq \mathbb{R}^{2n-1} \times S^1$.

For some recent results on the geometry of the components see [4], [11], [12]. It is useful to keep in mind that we have an algebraic map,

$$H : \mathrm{Rat}(n) \to \mathrm{Hank}(n)$$
$$g(s) \to H(g) = (h_{i+j-2})_{n\times n}$$

where
$$g(s) = \sum_{k=o}^{\infty} h_k/s^{k+1}$$

and $H(g)$ is a bilinear form of the Hankel type. That $H(g)$ is nondegenerate *iff* $g(s) \in \mathrm{Rat}(n)$ is a result that goes back to Cauchy-Hermite. Further, the Cauchy Index is given by,

$$\begin{aligned} I_{-\infty}^{\infty}(g) &= \sigma(H(g)) \\ &= \text{signature of } H(g) \ . \end{aligned} \tag{1.5}$$

This implies that the Cauchy index takes values in the set $\{-n,n+2,\ldots,n-2,n\}$.

We say that $g(s) \in \mathrm{Rat}(n)$ is of the stable and minimum-phase type if the poles and zeros of $g(s)$ lie in the *open* left half plane. We denote the set of such elements as $\mathrm{Rat}_{SM}(n) \subseteq \mathrm{Rat}(n)$. Further, the subset $\mathrm{Rat}(\mu,\nu) \cap \mathrm{Rat}_{SM}(n)$ will be denoted as $\mathrm{Rat}_{SM}(\mu,\nu)$, $(\mu+\nu=n)$. Since $g(s)$ is positive real *iff* $1/g(s)$ is [8], a positive real rational function has no poles or zeros in the right half plane. It is further convenient to treat separately those positive real rational functions that have some or all poles or zeros on the imaginary axis, since they may be viewed as limit points (they lie on the boundary). We thus have the set of *strictly positive real functions* denoted by,

$$\begin{aligned} \mathrm{Rat}_{pr}(n) \triangleq \{g(s) \in \mathrm{Rat}(n) : g(s) \text{ positive real and} \\ g(j\omega) + g(-j\omega) > 0, \ \ \omega \in \mathbb{R}\} \end{aligned} \tag{1.6}$$

We leave it to the reader to verify that $\mathrm{Rat}_{SM}(n)$ and $\mathrm{Rat}_{pr}(n) \subset \mathrm{Rat}_{SM}(n)$ are both open submanifolds of $\mathrm{Rat}(n)$. These manifolds will be the primary geometric objects of interest to us.

2. THE DEGREE 2 CASE. To fix ideas we first consider the case of degree 2 rational positive real functions. Given that

$$g(s) = \frac{as+b}{s^2+cs+d} \quad ,$$

it can be shown that, $g(s)$ is strict positive real (in our sense) iff,

$$\begin{cases} a > 0 \\ b > 0 \qquad d > 0 \qquad \text{and} \\ \frac{b}{a} \le c \qquad . \end{cases} \tag{2.1}$$

Now, any $g(s) \in \text{Rat}(2)$ has a representation [10], of the form

$$g(s) = \alpha \cdot \frac{[(s+\sigma)\cos(\theta) + \sin(\theta)]}{(s+\sigma)^2 + \nu} \tag{2.2}$$

$\nu \neq -\tan^2\theta$ ($\Leftrightarrow$ no common factors). Here $\alpha > 0$, $\sigma \in \mathbb{R}$, $\theta \in [0,2\pi)$ and $\nu \in \mathbb{R}$.

With this representation, the inequalities above reduce to

$$\begin{aligned} &\sigma > \tan(\theta) \\ &-\pi/2 < \theta < \pi/2 \\ &\sigma + \tan(\theta) > 0 \\ &\sigma^2 + \nu > 0 \end{aligned} \tag{2.3}$$

Now for $\text{Rat}_{pr}(1,1) = \text{Rat}_{pr}(2) \cap \text{Rat}(1,1)$ we have an additional inequality,

$$\nu > -\tan^2(\theta) \ . \tag{2.4}$$

Combining these inequalities, we have a parametrization of Rat(1,1) given by:

$$g_{II}(s) = \frac{e^{\alpha}[(s+e^{\beta})\cos(\theta) + \sin(\theta)]}{(s+e^{\beta})^2 + \ln(e^t + e^{-\tan^2\theta})}$$

where $\theta = -\tan^{-1}(e^{\beta}) + 2\tan^{-1}(e^{\beta})\frac{e^{\lambda}}{1+e^{\lambda}}$, $(\alpha,\beta,\lambda,t) \in \mathbb{R}^4$.

For $Rat_{pr}(2,0)$ the relevant inequalities are

$$\begin{cases} \sigma > \tan(\theta) \\ \sigma > -\tan(\theta) \\ \sigma^2 + \nu > 0 \\ \frac{-\pi}{2} < \theta < \frac{\pi}{2} \\ \nu < -\tan^2(\theta) \end{cases}$$

and we have a parametrization,

$$g_I(s) = \frac{e^{\alpha}[(s+e^{\beta})\cdot\cos(\theta)+\sin(\theta)]}{(s+e^{\beta})^2+(-e^{2\beta}+\frac{e^t}{1+e^t}(e^{2\beta}-\tan^2(\theta)))}$$

$$\theta = -\tan^{-1}(e^{\beta}) + 2\tan^{-1}(e^{\beta})\frac{e^{\lambda}}{1+e^{\lambda}}, \quad (\alpha,\beta,\lambda,t) \in \mathbb{R}^4 .$$

An alternative parametrization for this region is

$$g'_I(s) = \frac{e^{\alpha}}{s+e^{\beta}} + \frac{e^{\gamma}}{s+e^{\beta}+e^{\delta}}$$

$$(\alpha,\beta,\gamma,\delta) \in \mathbb{R}^4$$

This depends on the fact that sums of strict positive real functions are also strict positive real and the particular symmetry of $Rat_{pr}(2,0)$ (unrepeated poles in the open l.h.p with interlacing zero). There are no positive real functions in Rat(0,2) (there each element has negative residues). Thus we see that strict positive real rational functions of degree 2 are of two types distinguished by their respective Cauchy indices. The interesting consequence of the parametrizations indicated above is that both regions of strict passive systems are diffeomorphic to $\mathbb{R}^4$.

The direct treatment of the degree 2 case does not lend itself to generalization, since as the McMillan degree increases, so does the number of inequalities characterizing positive-realness. In the next section we show how some results on related classes of rational functions can be transferred to $Rat_{pr}(n)$.

3. SPECTRAL FACTORIZATION AND THE GEOMETRY OF $Rat_{SM}(n)$. Our aim in this section is to determine the number of homotopic equivalence classes of positive real functions. We shall see that the answer is quite intuitive in a network theoretic sense, since it has to do with the distinctive types of storage elements in a circuit. However, the actual proof seems to be best approached via the spectral factorization theorem and the geometry of the space of spectral factors. Specifically we begin by examining the space $Rat_{SM}(n)$ of proper rational functions of degree n with all poles and zeros constrained to lie in the open left half-plane.

The constraint on the disposition of poles and zeros makes the connectivity properties of $Rat_{SM}(n)$ quite different from those of $Rat(n)$. To understand this consider the following example:

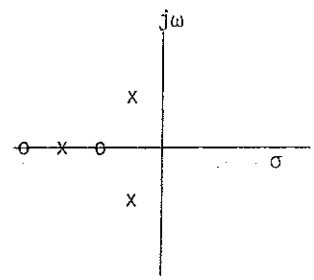


Fig. 1a

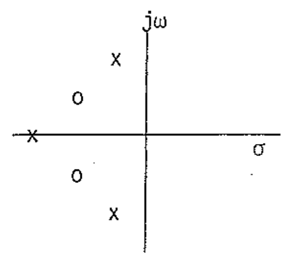
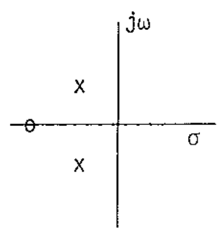


Fig. 1b

Since a pole-zero pattern determines a rational function up to scale factor, both patterns 1(a) and 1(b) can be thought of as representing rational functions in $Rat_{SM}(2,1)$. Pattern (a) can be deformed into pattern (b) continuously *only* by sending one of the zeros to ∞ and bringing it back to the other extreme, together with an adjacent zero and pairing off into the complex plane. In doing so, one has to pass through the r.h.p. thus leaving $Rat_{SM}(2,1)$. Thus we see that (a) cannot be deformed into (b) continuously within the stable and minimum-phase subset. It is not very hard to see that any other pole-zero pattern belonging to $Rat_{SM}(2,1)$ can be deformed into either (a) or (b). Thus $Rat_{SM}(2,1)$ has two connected components.

Difficulties arise even when there are no poles on the real axis. Consider the following example from $Rat_{SM}(1,1)$. (Figure 2)



$$g(s) = \frac{as + b}{s^2 + cs + d}$$

Figure 2

The Cauchy index $\sigma(g) = 0$. Since $\sigma(g) = -\sigma(-g)$ we note that $g(s)$ and $-g(s)$ both $\in Rat_{SM}(1,1)$. However $g(s)$ can be deformed into $-g(s)$ only be sending the zero to ∞ and bringing it back through the r.h.p., thereby violating the minimum-phase restriction. Thus $Rat_{SM}(1,1)$ has two components distinguished by the *sign of* a in

$$g(s) = \frac{as + b}{s^2 + cs + d} \quad .$$

A problem such as in the first example (Figure 1) does not arise when $\sigma(g) = \pm n$. Consider the example in Figure 3 showing a $g(s) \in Rat_{SM}(3,0)$. To have the correct index this is the standard pattern for $Rat_{SM}(3,0)$ and $Rat_{SM}(3,0)$ is connected. In general $Rat_{SM}(n,0)$ is connected,

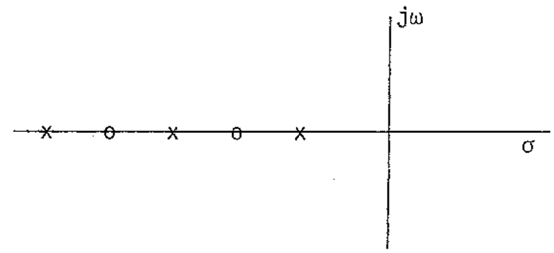


Figure 3

by an obvious extension of the above argument. Note that when $\sigma(g) \neq 0$, we need not consider the question of deforming $g(s)$ into $-g(s)$, since if $g(s) \in Rat_{SM}(p,q)$, $-g(s) \in Rat_{SM}(q,p)$ and $Rat(p,q)$ and $Rat(q,p)$ are distinct components anyway ($p \neq q$, *iff* $\sigma(g) \neq 0$).

For the general case of $Rat_{SM}(n)$, problems of the type posed in the first example arise whenever the index lies in the range $1 \leq |\sigma| \leq (n-2)$, since only in such cases we can have two standard patterns of interlacing poles and zeros on the real axis, which cannot be deformed into each other. (Note: the complex poles can be deformed arbitrarily and the complex zeros can all be sent to ∞). The standard patterns are:

a) zero, pole, zero,..., pole, zero

poles = $|\sigma|$

zeros = $|\sigma| + 1$

b) pole, zero, pole,..., zero, pole

poles = $|\sigma|$

zeros = $|\sigma| - 1$

This depends on the fact that a typical element of $Rat(p,q)$ has $(n-1)$ zeros and an even number of them are complex. By the same argument as in the first example, we see that pattern (a) cannot be deformed into pattern (b) within the class of stable and minimum-phase systems, because the zero closest to the origin cannot be removed without either violating the common factor condition or the minimum-phase condition,--i.e., it is a trapped zero. Further, if n is even then $\mathrm{Rat}(\frac{n}{2}, \frac{n}{2})$ has index zero and by an extension of the argument made in the second example, it has two path-components. The generalization of example 3 has already been made. Collecting together the above arguments we have the following theorem (somewhat similar to Theorem 2 of [10]).

THEOREM 1. $\mathrm{Rat}_{SM}(p,q)$ *has two connected components if* $|p - q| \leq (n-2)$, *where* $n = p+q$, *and* $\mathrm{Rat}_{SM}(n,0)$ *and* $\mathrm{Rat}_{SM}(0,n)$ *are respectively connected. Thus* $\mathrm{Rat}_{SM}(n)$ *has* $2n$ *connected components.* ⌑

REMARK. In the proof outlined above we have not explicitly described the homotopies, partly since the arguments using deformation of pole-zero patterns are direct. This of course depends on the continuous dependence of zeros of polynomials on their coefficients (c.f. [13], Thm. 1,4).

We pass from Theorem 1 to a result on positive real functions using spectral factorization. Consider the set of rational functions with special symmetry,

$$\mathrm{Rat}_{sd}(2n) \triangleq \{\phi(s) \in \mathrm{Rat}(2n) : \phi(s) = \phi(-s),$$

$$0 < \phi(i\omega) < \infty, \text{ for } \omega \in \mathbb{R}, \text{ and}$$

$$\int_{-\infty}^{\infty} \phi(i\omega)d\omega = a < \infty \}$$

Any element of $\mathrm{Rat}_{sd}(2n)$ has a certain quadrantal symmetry of the pole-zero pattern and appears as the spectral density function of a second order stationary stochastic process. With the conditions in the definition, no poles or zeros are admitted on the imaginary axis. Further, since

$$\sigma(\phi(-s)) = -\sigma(\phi(s))$$

all $\phi(s) \in \mathrm{Rat}_{sd}(2n)$ have Cauchy index zero. In other words, $\mathrm{Rat}_{sd}(2n) \subset \mathrm{Rat}(n,n)$. We have the standard result [14].

THEOREM 2. (spectral facotrization): *Let* $\phi(s) \in \mathrm{Rat}_{sd}(2n)$. *Then in terms of poles and zeros, we have the representation,*

$$\phi(s) = \frac{c_0 \prod_j (s+q_j)(s-q_j) \prod_i [(s+\mu_i)^2 + \lambda_i^2][(-s+\mu_i)^2+\lambda_i^2]}{\prod_j (s+p_j)(s-p_j) \prod_i [(s+\sigma_i)^2+\omega_i^2][(-s+\sigma_i^2)+\omega_i^2]}$$

without common factors and

$$c_0 > 0, \quad q_j > 0, \quad \mu_i > 0, \quad \lambda_i > 0$$

$$p_j > 0, \quad \sigma_i > 0, \quad \omega_i > 0 \quad .$$

Hence we have the factorization $\phi(s) = g(s)g(-s)$ *where* $g(s) \in \mathrm{Rat}_{SM}(n)$ *is given by*

$$g(s) = \pm \sqrt{c_0}\, \frac{\prod_j (s+q_j) \prod_i [(s+\mu_i)^2 + \lambda_i^2]}{\prod_j (s+p_j) \prod_i [(s+\sigma_i)^2 + \omega_i^2]} \quad . \qquad \Pi$$

From the spectral factorization theorem above, any $\phi(s) \in \mathrm{Rat}_{sd}(2n)$ has two valid l.h.p. factors $g(s) \in \mathrm{Rat}_{SM}(p,q)$ and $-g(s) \in \mathrm{Rat}_{SM}(q,p)$. If $p-q=0$, than both $g(s)$ and $-g(s)$ have index = 0. However, they lie in distinct components characterized by the sign of the scale factor k in

$$g(s) = k\, \frac{\prod_j (s+q_j) \prod_i [(s+\mu_i)^2 + \lambda_i^2]}{\prod_j (s+p_j) \prod_i [(s+\sigma_i)^2 + \omega_i^2]}$$

To eliminate ambiguity we consider only l.h.p. factors with Cauchy index ≥ 0 and when index = 0, only the factor $g(s)$ with positive scale factor. For the course of the present discussion, we call such factors *canonical factors*. We denote the set of canonical (l.h.p.) factors as $\mathrm{Rat}^{+}_{SM}(n)$. It has n connected components. A basic result on the geometry of

$\mathrm{Rat}_{sd}(2n)$ is the following.

THEOREM 3. $\mathrm{Rat}_{sd}(2n)$ *has* n *connected components. Hence* $\mathrm{Rat}_{pr}(n)$ *has* n *connected components.*

Proof. Consider the map,

$$\psi : \mathrm{Rat}^{+}_{SM}(n) \to \mathrm{Rat}_{sd}(2n)$$

$$g(s) \to \phi(s) = g(s) \cdot g(-s)$$

That ψ is one-to-one and onto follows from the spectral factorization theorem and the definition of $\mathrm{Rat}_{sd}(2n)$. Once again from the continuous dependence of zeros of a polynomial on its coefficients, it follows that ψ is a homeomorphism. Since the number of connected components is a topological invariant and using Theorem 1, we have the statement about $\mathrm{Rat}_{sd}(2n)$.

Now, any $\phi(s) \in \mathrm{Rat}_{sd}(2n)$ has a unique partial-fraction decomposition, $\phi(s) = [\phi(s)]_+ + [\phi(s)]_-$ where $[\phi(s)]_+$ is a positive real function (c.f. [8]) and $[\phi(s)]_- = [\phi(-s)]_+$. Since by definition $\phi(s) \in \mathrm{Rat}_{sd}(2n)$ has no poles or zeros on the imaginary axis, neither does $[\phi(s)]_+$. In fact $[\phi(s)]_+ \in \mathrm{Rat}_{pr}(n)$ the set of *strict positive real functions*. Consider the map:

$$[\quad]_+ : \mathrm{Rat}_{sd}(2n) \to \mathrm{Rat}_{pr}(n)$$

$$\phi(s) \to [\phi(s)]_+$$

Using the uniqueness of the partial fraction decomposition and the definition of $\mathrm{Rat}_{pr}(n)$ we see that the map $[\quad]_+$ is one-to-one and onto. It is *bicontinuous* for the same reasons as ψ is. Once again we have from topological invariance,

$$n = \#\text{ components of } \mathrm{Rat}_{pr}(n)$$
$$= \#\text{ components of } \mathrm{Rat}_{sd}(2n) = \#\text{ components of } \mathrm{Rat}^{+}_{SM}(n). \quad ⌑$$

Denoting as $\mathrm{Rat}_{pr}(p,q) = \mathrm{Rat}_{pr}(n) \cap \mathrm{Rat}(p,q)$ each of the n connected-components of $\mathrm{Rat}_{pr}(n)$ is distinguished by its Cauchy index. Since there are $(n+1)$ possible values of the Cauchy index it is clear that one of them is disallowed. We see below which one it is. Firstly,

$$\mathrm{Rat}_{pr}(n,0) = \mathrm{Rat}_{SM}(n,0)$$

This is a consequence of the fact that $\mathrm{Rat}_{SM}(n,0)$ has the global parametrization,

$$g(s) = \frac{e^{\alpha_1}}{s+e^{\lambda_1}} + \frac{e^{\alpha_2}}{s+e^{\lambda_1}+e^{\lambda_2}} + \dots + \frac{e^{\alpha_n}}{s+e^{\lambda_1}+\dots+e^{\lambda_n}}$$

$$(\alpha_1,\alpha_2,\dots,\alpha_n,\lambda_1,\lambda_2,\dots,\lambda_n) \in \mathbb{R}^{2n}$$

where each term on the right is strict positive real (also since $Z_1(s) + Z_2(s)$ is positive real whenever $Z_1(s)$ and $Z_2(s)$ are). Now if $g(s) \in \mathrm{Rat}_{SM}(0,n)$ is positive real, then $-g(s) \in \mathrm{Rat}_{SM}(n,0)$ is not positive real. But we note that $\mathrm{Rat}_{SM}(n,0) = \mathrm{Rat}_{pr}(n,0)$. Thus, $\mathrm{Rat}_{SM}(0,n) \cap \mathrm{Rat}_{pr}(0,n)$ = the empty set. But $\mathrm{Rat}_{pr}(0,n) \subset \mathrm{Rat}_{SM}(0,n)$ by definition of positive reality. Hence $\mathrm{Rat}_{pr}(0,n)$ = empty set.

Thus there are no proper rational (strict) positive real functions of index $(-n)$. This classification of passive systems can be viewed as a classification in terms of types of storage elements, since in the setting of R-L-C networks, the Cauchy index = # capacitors - # inductors, and the rational function of interest is a network impedance at the driving point [15].

REMARK 1. It is quite intuitive that networks having different numbers of capacitors and inductors (but the same total) should not be continuously deformable into each other. What is perhaps surprising about Theorem 2 is that there are no additional obstructions to continuous deformations.

REMARK 2. In practice, the discrete-time setting (functions positive real with respect to the unit circle) is often applicable and following the methods of this section one would be interested in the spaces

$$\mathrm{DRat}_{SM}(n) \triangleq \{g(z) \in \mathrm{Rat}(n) : \text{poles and zeros of } g(z) \text{ lie in the disk } \{z : |z| < 1\}\}$$

and

$$\mathrm{DRat}_{sd}(2n) \triangleq \{\phi(z) \in \mathrm{Rat}(2n) : \phi(z) = \phi(z^{-1}) \quad 0 < \phi(e^{j\omega}) < \infty \text{ for } \omega \in (0, 2\pi], \text{ and } 0 < \int_{-\pi}^{\pi} \phi(e^{j\omega}) d\omega \leq a < \infty \}$$

The situation is complicated by the boundedness of the domain in which the zeros and poles are constrained to lie. We do not pursue this here further than to state the following result (see [16] for proof. Also see Theorem 2 in [10]).

THEOREM 4. $\mathrm{DRat}_{SM}(n)$ *has* $n(n-1)$ *connected components. Rational functions with distinct numbers of zeros cannot be deformed into each other.* ⌘

In the following sections we examine more closely the various properties of rational positive real functions and how these relate to the geometry of the space $\mathrm{Rat}_{pr}(n)$.

4. DARLINGTON SYNTEHSIS AND A COVERING SPACE FOR POSITIVE REAL FUNCTIONS. At the end of section 3 we determined the connectivity of the space $\mathrm{Rat}_{pr}(n)$ of strict positive real functions. Also we had a global parametrization of the degree 2 case in section 2. In this section we consider the general situation. Our aim is to show that the classical result in network synthesis due to Sydney Darlington [17] leads to a geometric theorem on $\mathrm{Rat}_{pr}(n)$. It is important to note that a condition of *compactness* of two ports plays a crucial role. Here we state a modern version of Darlington's theorem due to Brockett [18] (also [22]).

THEOREM 5. (Darlington-Brockett). *Let* $g(s)$ *be a positive real function. Then* $g(s)$ *has a minimal realization of the following form.*

$$\dot{z} = (S - pp')z + qu$$
$$y = \langle q,z \rangle$$

where $S = -S'$ *is real and* $q,p \in \mathbb{R}^n$. ⌑

REMARK. This is a state-space version of the classical Darlington synthesis which says that any positive real function can be realized by terminating a lossless 2-port by a (1 - ohm) resistor. To see the relationship, consider the two-input, two-output system (i.e., 2-port) below:

$$\dot{x} = Sx + qu + pv$$
$$y_1 = \langle q,x \rangle \qquad S = -S' \tag{4.2}$$
$$y_2 = \langle p,x \rangle$$

The realization in Theorem 5 corresponds to the feedback $u_2 = -y_2$ followed by the identification $y = y_1$. Further (4.2) is lossless. The transfer function of the 2-port is

$$Z(s) = \begin{pmatrix} z_{11}(x) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{pmatrix} = \begin{pmatrix} \langle q,(sI-S)^{-1}q \rangle & \langle q,(sI-S)^{-1}p \rangle \\ \langle p,(sI-S)^{-1}q \rangle & \langle p,(sI-S)^{-1}p \rangle \end{pmatrix}$$

The McMillan degree $\delta[g(s)] = \delta[Z(s)] = n$, i.e., if $[S - pp',q,q']$ minimal, then so is $[S,(q|p), (\frac{q'}{p'})]$. This is because, $[S - pp',q,q']$ minimal $\Rightarrow [S - pp',(q|p), (\frac{q'}{p'})]$ (additional controls and observations do not lower McMillan degree). But $S - pp' = S - (q|p) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (\frac{q'}{p'}) = S - (q|p)K(\frac{q'}{p'})$. Thus $[S - pp',(q|p),(\frac{q'}{p'})]$ is in the same (output) feedback equivalence class as $[S,(q|p),(\frac{q'}{p'})]$ and we know that systems in the same feedback equivalence class have the same McMillan degree [14].

We have the following useful characterization of Darlington realizations.

THEOREM 6. *Let* $g(s)$ *be a given positive real function and let* $Z(s)$ *be as in Theorem 4, such that*

$$g(s) = \langle q, (sI - S + pp')^{-1} q\rangle; \quad S = -S' .$$

Then $Z(s)$ *is uniquely determined by* $g(s)$ *upto a factor of* ± 1 *in the coupling parameters* z_{12} *and* z_{21}.

Proof. Let $[S_1 - p_1p_1', q_1, q_1']$ and $[S_2 - p_2p_2', q_2, q_2']$ be two (minimal) Darlington realizations of the same $g(s)$. Hence $S_1 = -S_1'$; $S_2 = -S_2'$. Then by the state space isomorphism theorem, there exists a unique nonsingular matrix P such that

$$P(S_1 - p_1p_1')P^{-1} = S_2 - p_2p_2'$$

$$Pq_1 = q_2$$

$$q_1' P^{-1} = q_2'$$

Claim: P belongs to $O(n)$ the group of $n \times n$ orthogonal matrices i.e., $PP' = I$.

We give a simple geometrical argument for this. The set of Darlington realizations $[S - pp', q, q']$ is clearly a manifold of dimension $\frac{n(n+3)}{2}$. The subgroup of $G\ell(n)$ acting on this manifold according to the state-space isomorphism theorem acts freely (by uniqueness of P) and hence must be of dimension $\frac{n(n-1)}{2}$ so that the dimension of the orbit-space $= \frac{n(n+3)}{2} - \frac{n(n-1)}{2} = 2n =$ dimension of the manifold of positive real functions. Further since the map, $[S - pp', q, q'] \to [P(S - pp')P', Pq, q'P']$ where $P \in O(n)$ carries Darlington realizations into Darlington realizations, our group of isomorphisms must contain $O(n)$. But $O(n)$ is itself of dimension $\frac{n(n-1)}{2}$. Hence the group is $O(n)$ and our claim is verified.

Note that $P(S - pp')P' = PSP' - (Pp)(Pp)'$ where $S = -S'$ and $P \in O(n)$ implies that PSP' is skew-symmetric. Hence, for

$PP' = I,$

$$P(S_1 - p_1p_1')P' = S_2 - p_2p_2'$$
$$Pq_1 = q_2 \qquad S_1 = -S_1'$$
$$q_1P' = q_2' \qquad S_2 = -S_2'$$
$$\Rightarrow \qquad S_2 = PS_1P' \; ; \; (Pp_1)(Pp_1)' = p_2p_2'$$

Hence $p_2 = \pm Pp_1$. Thus if $[S - pp', q, q']$ is a Darlington realization of a given positive real rational function, then the 2-ports associated with it via Theorem 5 are given by the triples,

$$[PSP', \; (Pq|\pm Pp), \; \begin{pmatrix} q'P' \\ \pm p'P' \end{pmatrix}]$$

where $P \in O(n)$. It is easily verified that they have the transfer function

$$Z(s) = \begin{pmatrix} z_{11}(s) & \pm z_{12}(s) \\ \pm z_{21}(s) & z_{22}(s) \end{pmatrix}$$

where

$$z_{11}(s) = \langle q, (sI - S)^{-1}q\rangle \; ; \quad z_{21}(s) = \langle p, (sI - S)^{-1}q\rangle$$

$$z_{12}(s) = \langle q, (sI - S)^{-1}p\rangle \; ; \quad z_{22}(s) = \langle p, (sI - S)^{-1}p\rangle$$

Hence, the theorem. ♯

Before we proceed to interpret this result, several remarks are in order.

1. $z_{11}(s)$ and $z_{22}(s)$ are lossless 1-ports. Further, $\mathrm{Re}[z_{12}(j\omega) + z_{21}(j\omega)] = 0$ for $\omega \in \mathbb{R}$
2. The poles and zeros of $z_{11}(s)$ and $z_{22}(s)$ are simple and interlace on the imaginary axis. Further, the residues at these poles are ≥ 0.
3. $Z(s)$ has the following partial-fraction expansion

$$Z(s) = \begin{cases} \frac{1}{s} R_o + \sum_{k=1}^{\frac{n-1}{2}} \left(\frac{1}{s+j\lambda_k} R_k + \frac{1}{s-j\lambda_k} \bar{R}_k\right) & \lambda_k \neq \lambda_\ell \text{ if } k \neq \ell; \ n \text{ \underline{odd}} \\ \sum_{k=1}^{n/2} \left(\frac{1}{s+j\lambda_k} R_k + \frac{1}{s-j\lambda_k} \bar{R}_k\right) & \lambda_k \neq \lambda_\ell \text{ if } k \neq \ell; \ n \text{ \underline{even}} \end{cases}$$

where

$$R_k = \begin{pmatrix} r_{11}^k & r_{12}^k \\ r_{21}^k & r_{22}^k \end{pmatrix}$$

is a residue matrix satisfying,

$$r_{11}^k = \bar{r}_{11}^k \geq 0$$

$$r_{22}^k = \bar{r}_{22}^k \geq 0$$

Further, R_o is real symmetric and R_k is Hermitian (i.e., $r_{12}^k = \bar{r}_{21}^k$).

4. The residue matrices are of rank 1. This follows from the fact that McMillan degree $\delta[Z(s)]$ = sum of the ranks of residue matrices, and we noted in the remark after Theorem 5 that $\delta[Z(s)] = \delta[g(s)] = n$, if $g(s)$ does not have common factors. As a consequence of this rank condition we have the *residue condition*,

$$r_{11}^k \, r_{22}^k - r_{12}^k \, r_{21}^k = 0, \quad \text{for each } k.$$

2 ports which satisfy this condition are known as *compact* 2 ports (see [19]) for an interpretation of compactness as an *external* symmetry condition).

5. In Theorem 5 and Theorem 6,

$$g(s) = z_{11} - \frac{z_{12}\, z_{21}}{1 + z_{22}}$$

and $g(s)$ is positive real in the usual sense. When $g(s)$ is lossless positive real (i.e., $p = 0$), Theorem 5 associates a unique 2-port,

$$Z(s) = \begin{pmatrix} g(s) & 0 \\ 0 & 0 \end{pmatrix}$$

If $g(s)$ is not lossless, then by Theorem 6 there are exactly two *compact lossless* 2-ports of degree n mapped into the same positive real function.

6. Even if the McMillan degree of the 2-port $Z(s)$ is n, it is possible for $g(s) = z_{11} - \frac{z_{12}\, z_{21}}{1 + z_{22}}$, to have common factors. For example, if $g(s) = \frac{as + b}{s^2 + cs + d}$ is positive real then it can be realized (by Darlington synthesis) from a pair of compact Hermitian 2-ports,

$$Z(s) = \frac{1}{s^2 + d} \begin{pmatrix} as & sf + g \\ sf - g & cs \end{pmatrix}$$

where $f = \pm \sqrt{ac - b}$ and $g = \pm \sqrt{bd}$. As $bf^2 \to da^2$, the McMillan degree of Z is still 2 but $g(s) \to$ pole-zero cancellation. We can still use the properties of the map $\psi : Z(s) \to g(s)$ as below.

Let $PR(n) = \bigcup_{k=1}^{n} Rat_{pr}(k)$. Let $\tilde{KL}(n)$ denote the space of all compact lossless Hermitian 2-parts of McMillan degree n. The map defined in Theorem 5 is denoted as

$$\psi : \tilde{KL}(n) \to \text{positive real functions}$$
$$Z(s) \to g(s) = z_{11}(s) - \frac{z_{12}(s)\, z_{21}(s)}{1 + z_{22}(s)}$$

We denote as $KL(n) \subset \tilde{KL}(n)$, the inverse image

$$KL(n) = \psi^{-1}[PR(n)] ,$$

and the restriction $\psi|_{KL(n)}$ as ψ_d. Then, since $PR(n)$ does not contain any lossless positive real functions,

$$\psi_d : KL(n) \to PR(n)$$

is a *two-to-one* map. Using the compactness condition one can show that each residue matrix is of rank 1 with positive diagonal elements. We thus have a parametrization of $KL(n)$ via residue matrices,

$$r_{11}{}^k \in (0,\infty) ;$$

$$r_{22}{}^k \in (0,\infty) ;$$

$$r_{12}{}^k = \bar{r}_{21}{}^k = e^{i\theta} \sqrt{r_{11}{}^k r_{22}{}^k}$$

$$\theta \in S^1$$

and letting $\lambda_k \in (0,\infty)$, we see that

$$KL(n) \approx \begin{cases} \mathbb{R}^{\frac{3n}{2}} \times (S^1)^{\frac{n}{2}} & ; n \text{ even} \\ \mathbb{R}^{\frac{3n+1}{2}} \times (S^1)^{\frac{n-1}{2}} & ; n \text{ odd.} \end{cases}$$

Further, from Theorem 6 and the remarks following it we have seen that for each $g(s) \in PR(n)$ (none of which is lossless) there are two members of $KL(n)$, $Z^{(1)}(s)$ and $Z^{(2)}(s)$ differing in the signs of the coupling parameters z_{12} and z_{21}. It is easy to see that $Z^{(1)}(s)$ and $Z^{(2)}(s)$ correspond to *antipodal points* on the circles in the parametrization above. It is thus possible to treat $PR(n)$ as the quotient space.

$$KL(n)/\sim$$

where the equivalence relation "$\sim$" is simply the identification of antipodal points on a circle. Of course $S^1/\sim = RP^1$ the

1-dimensional real projective space $\simeq S^1$ and we have,

THEOREM 7. *The set of strict positive real functions of degree* $\leq n$ *has the global structure,*

$$PR(n) \simeq \begin{cases} (S^1)^{\frac{n}{2}} \times \mathbb{R}^{\frac{3n}{2}} & ; \ n \ \text{even} \\ (S^1)^{\frac{n-1}{2}} \times \mathbb{R}^{\frac{3n+1}{2}} & ; \ n \ \text{odd}. \end{cases} \qquad \square$$

This result can be interpretated as the construction of a *covering space* for $PR(n)$. Recall [20] that a covering space for a topological space X is a pair $(\tilde{X},\pi)$ consisting of a space $\tilde{X}$ and a (continuous) projection $\pi : \tilde{X} \to X$ such that every point $x \in X$ is contained in a (small enough) neighborhood U satisfying:

$$\begin{array}{ccc} U \times \pi^{-1}(x) & \xrightarrow{\ \simeq\ } & \pi^{-1}(U) \subseteq \tilde{X} \\ & {\scriptstyle pr_1} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & U & \end{array}$$

Further the fiber $\pi^{-1}(x)$ has to be discrete. If the cardinality of $\pi^{-1}(x)$ is $= n$, then we say that $\tilde{X}$ is an n-fold cover of X.

In the present case $KL(n)$ is simply a 2-fold cover of $PR(n)$. The exclusion of elements of the type

$$Z(s) = \begin{pmatrix} g(s) & 0 \\ 0 & 0 \end{pmatrix}$$

from $KL(n)$ thus guarantees the evenness of the cover.

5. FINAL REMARKS. In this paper we have investigated the geometry of linear passive systems and related classes of systems. We expect some of these results to be applicable to problems of approximate covariance generation. Using some of the

work of Oono and Yasuura [21] it is possible to generalize the results of section 4 to multivariable passive systems.

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