

Geometric Phases, Anholonomy, and Optimal Movement

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Abstract

In the search for useful strategies for movement of robotic systems (e.g. manipulators, platforms) in constrained environments (e.g. in space, underwater), there appear to be new principles emerging from a deeper geometric understanding of optimal movements of nonholonomically constrained systems. In our work, we have exploited some new formulas for geometric phase shifts to derive effective control strategies. The theory of connections in principal bundles provides the proper framework for questions of the type addressed in this paper. We outline the essentials of this theory. A related optimal control problem and its localizations are also considered.

1 Background (Anholonomy)

This paper is a continuation of [6]. Consider a robot (kinematic chain of n rigid bodies) floating in zero gravity. For convenience, assume that the bodies (and the assembly) are planar and the assembly is initially at rest (angular momentum $\mu = 0$) as in Figure 1. Suppose the joint angles are varied continuously and brought back to rest in a prescribed manner. There will be a net displacement (say of body 1) from its absolute initial orientation. This phase shift is given by an integral formula [6]

$$\Delta q_1 = - \int_{\gamma} \frac{\mathbf{e} \cdot \mathbf{J} \mathbf{M}}{\mathbf{e} \cdot \mathbf{J} \mathbf{e}} d\theta \quad (1)$$

*Supported in part by the National Science Foundation's Engineering Research Centers Program: NSF CDR 8803012, and by the AFOSR URI Program under grant AFOSR-90-0105

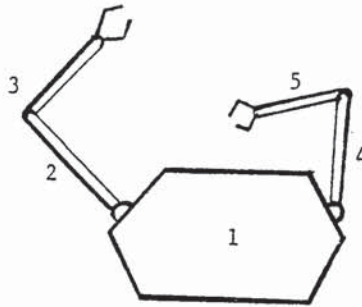


Figure 1: Planar kinematic chain

where $\mathbf{e} = (1, 1, \dots, 1)$, M is an $n \times (n-1)$ matrix given by

$$M_{ij} = \begin{cases} 0, & i = 1 \\ 1, & i > j \\ 0 & \text{otherwise,} \end{cases}$$

and \mathbf{J} denotes the $n \times n$ kinetic energy quadratic form associated to the planar n -body system. See [15] for explicit formulas for \mathbf{J} . In formula (1) the curve γ is in the $n-1$ dimensional space of joint variables (a torus in the case of revolute joints, ignoring angle limits). A key observation is that the integral (1) depends on γ but *not* on how fast it is traversed. Hence we refer to Δq_1 as *geometric phase*. For $n > 2$, a closed curve γ can be the boundary of a smooth surface in the space of joint angles. Then, by Stokes' theorem, the geometric phase Δq_1 is given by,

$$\Delta q_1 = - \iint_{\Gamma} d\left(\frac{\mathbf{e} \cdot \mathbf{J} \mathbf{M}}{\mathbf{e} \cdot \mathbf{J} \mathbf{e}}\right) d\theta, \quad (2)$$

Where Γ is any surface in the space of joint angles with boundary $\partial\Gamma = \gamma$. What we have discussed so far is an instance of the abstract setting consisting of:

(a). a simple mechanical system with symmetry (Q, K, V, G) with configuration space Q , kinetic energy quadratic form (or riemannian metric on Q) given by K , a Lie group G acting on Q leaving invariant K and potential energy V ;

(b). a principal G -bundle $(Q, Q/G, G)$ where the space $S = Q/G$ is known as the shape space;

(c). controls (forces) acting on (Q, K, V, G) , also leaving invariant the conserved momentum map $J^\sharp : TQ \rightarrow \mathcal{G}^*$ the dual of the Lie algebra of G , associated to the free hamiltonian system with energy $K + V$.

The map J^\sharp is given explicitly by the formula

$$J^\sharp(w_q)\xi = (K^\flat w_q)(\xi_Q(q)) \quad (3)$$

where $\xi \in \mathcal{G}$ the Lie algebra of G , K^\flat is the Legendre transform and ξ_Q is the infinitesimal generator (vector field on Q) associated to ξ . Let \mathbf{I}_q denote the symmetric bilinear form on \mathcal{G} ,

$$\mathbf{I}_q(\xi, \eta) = K(\xi_Q(q), \eta_Q(q)). \quad (4)$$

Let $\mathbf{I}_q^\flat : \mathcal{G} \rightarrow \mathcal{G}^*$ be the corresponding pairing. Then we have a vertical-horizontal splitting, of the tangent bundle TQ ,

$$\begin{aligned} TQ_q &= (Vert)_q \oplus (Hor)_q \\ w_q &= ((\mathbf{I}_q^\flat)^{-1}\mu)_Q(q) + (w_q - ((\mathbf{I}_q^\flat)^{-1}\mu)_Q(q)) \end{aligned}$$

where $\mu = J^\sharp(w_q)$. This splitting has the equivariance property with respect to the G -action on Q and defines a principal connection [12]. The connection appears to be originally due to Smale and Kummer [8].

For the planar n -body problem, $Q = T^n$, $S = T^{n-1}$ is the joint-space, and the expression $-\frac{e \cdot \mathbf{J}M}{e \cdot \mathbf{J}e} d\theta$ is simply the connection 1-form. The essence of the *anholonomy* lies in the fact that the Lie bracket of two horizontal vectors is *not* horizontal. Equivalently the curvature form (integrand in (2)) is typically non-vanishing on the shape space. This is what gives rise to the geometric phase shift.

Remark 1. Formulas (1) and (2) can be used to plan movements in shape space with prescribed phase shifts. A basic strategy is to use a family of standard loops γ in shape space. Regions of high curvature lead to increased phase shifts. Figure 2 represents the curvature form for a planar 3-body problem. In [5], there is a discussion of a motion planner using look up tables, interpolation and the formula (1). For a typical re-orientation maneuver from this planner, see Figure 3.

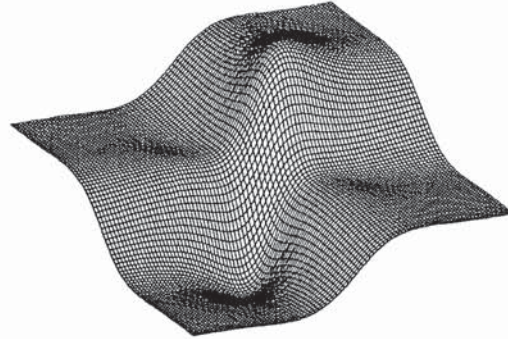


Figure 2: Curvature for 3-body problem

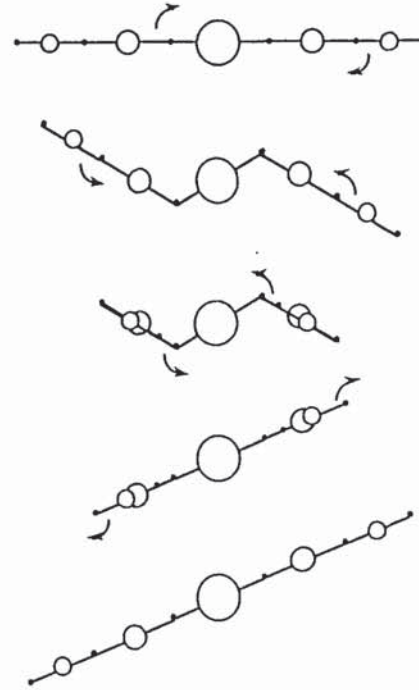


Figure 3: Reorientation through shape change

Remark 2. In [8] and [9], formulas analogous to (1) are given for a simple rigid body in 3 dimensions with nonzero angular momentum. For a related formula applicable to gyrostats see [1]. Analogous formulas for coupled rigid body systems and systems with flexible attachments are known to the authors and may be found in [16].

2 Optimal Movement

In the pioneering work of Shapere and Wilczek [13] and [14], a principal motivation was to solve (approximately) the problem of optimal shape change to achieve a prescribed holonomy. The physical setting was that of a planar deformable body immersed in a fluid at very low Reynolds number. Efficient self-propulsion was the goal. Here we consider the problem of optimal shape change with prescribed geometric phase shift for a kinematic chain.

For an n -body chain as section 1, with revolute joints, let $q = (q_1, \dots, q_n)^T$ denote the n -tuple of absolute orientations. Then, with angular momentum $\mu = 0$, $I =$ identity matrix,

$$\dot{q} = (I - \frac{ee^T \mathbf{J}}{e \cdot \mathbf{J}e})M\dot{\theta}. \quad (5)$$

Treating $v = \dot{\theta}$ the “shape velocity” as a control, a problem of interest is to determine a shape change sequence γ returning to the initial shape and giving rise to a prescribed phase shift while minimizing the cost functional

$$\eta = \int_{\gamma} \|\dot{\theta}\|^2 dt. \quad (6)$$

Montgomery refers to problems of this type as isoholonomy problems and obtains among other things, the hamiltonian equations governing γ [10]. Pontryagin’s maximum principle leads to the same conditions,

$$\begin{aligned} \dot{q} &= D(q)D^T(q)p \\ \dot{p} &= -\frac{\partial}{\partial q}(\frac{1}{2}p^T D(q)D^T(q)p) \\ v &= D^T(q)p \end{aligned} \quad (7)$$

where,

$$D = (I - \frac{ee^T \mathbf{J}}{e \cdot \mathbf{J}e}).$$

In general (7) is not explicitly solvable and numerical methods are needed to attack this boundary value problem. Montgomery considers examples where explicit solutions exist.

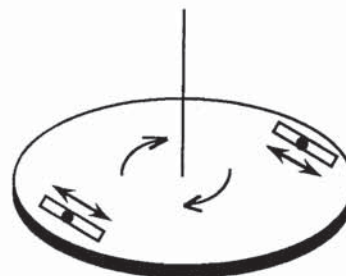


Figure 4: Rigid body with two point masses

3 Localization

Geometric phase shifts are a form of secular drift. They can be therefore accumulated by repeatedly traversing the same path γ in shape space. In particular, small loops γ repeated many times can lead to large phase drifts. A small loop γ corresponds to a *localized* change of shape. By a *localization* of the *optimal control problem* (5)-(6), we mean replacing (5) by a suitable normal form, e.g. linearization about a reference shape.

In Brockett’s papers [2] and [3] a model problem of the following form is considered:

$$\min \int_0^1 (u^2 + v^2) dt \quad (8)$$

subject to

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= xv - yu \end{aligned} \quad (9)$$

and boundary conditions $x(0) = x(1)$, $y(0) = y(1)$ and $z(0)$ and $z(1)$ specified. This optimal control problem is a normal form (linear). As Brockett notes in [2] [3], the difference $z(1) - z(0)$ is the area of a Lissajous figure. One might thus call this the *area normal form*. The problem is explicitly solvable as shown by Brockett.

Now consider the following set-up of a planar 3-body system with a central body and two point masses confined to move *linearly* along guideways as in Figure 4. If the masses are identical and the guideways are parallel, equidistant from the center of mass of the central body, then the localization of the corresponding isoholonomy problem takes the form

$$\min \int_0^1 (u^2 + v^2) dt \quad (10)$$

subject to

$$\begin{aligned}\dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= x^2v - y^2u\end{aligned}\quad (11)$$

for given conditions $x(0) = x(1)$, $y(0) = y(1)$ and $z(0), z(1)$ specified. We refer to (10)-(11) as an *area moment* normal form. The differential equations for geodesics are:

$$\begin{aligned}\ddot{x} - \lambda(x+y)\dot{y} &= 0 \\ \ddot{y} + \lambda(x+y)\dot{x} &= 0 \\ \ddot{z} + 2(y-x)\dot{x}\dot{y} + \lambda(x+y)(x^2\dot{x} + y^2\dot{y}) &= 0.\end{aligned}\quad (12)$$

Observe that there is a first integral $\dot{x}^2 + \dot{y}^2$. Here λ is a Lagrange multiplier. Letting $w = x + y$, it is easy to see that, for a constant c ,

$$\ddot{w} + \lambda w(c + \frac{\lambda}{2}w^2) = 0\quad (13)$$

the equation for a quartic oscillator, solvable by elliptic functions. It follows that the area-moment normal form (10)-(11) is also explicitly solvable. This is also known to Brockett and L.Dai [4]. More details on this class of problems viewed from the context of Nilpotent Lie groups may be found in [7].

4 Final Remarks

We have outlined the role of geometric phases and related optimal control problems in the maneuvers of floating kinematic chains. Lack of holonomy is the key feature. Other aspects of nonholonomic mechanics as it relates to motion planning have been investigated by a number of authors including S. Sastry, Z. Li, J. Canny, and their collaborators and by R. Brockett and D. Montana. For a list of relevant references see the paper of Murray and Sastry [11].

In a forthcoming paper [7] we investigate the differential geometry of localized optimal control problems formulated in the present paper.

We would like to thank R.W. Brockett and W. Dayawansa for many helpful discussions.

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