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# Control Problems on Principal Bundles and Nonholonomic Mechanics

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ABSTRACT. In this paper we investigate some problems of open-loop control defined on principal bundles with connections. We are interested in the explicit solvability of related optimal control problems from a hamiltonian point of view. The motivations for these problems arise in the study of certain nonholonomic mechanical systems with symmetry. Here we focus on a model problem admitting $SO(3)$ as the group of symmetries. The physical setting of this model problem corresponds to a rigid body with two oscillators. Under suitable hypotheses and approximations, symplectic and Poisson reduction are applied.

## 1. Introduction

Precise nonlinear dynamic models have had a significant role in the analysis, simulation and control of a variety of mechanical systems. The use of global geometric methods, including concepts based on symplectic geometry, symmetry groups and reduction, leads to valuable insight into the qualitative properties of such models. In the formulation of (constrained) motion planning problems (e.g. for space-based robots), a very powerful tool is derived from the theory of geometric phases. It is noteworthy that geometric phases have recently been the subject of intense study in a variety of areas of classical and quantum physics. In the search for a common mathematical framework underlying such applications in modern physics and in the classical mechanical problems of interest to us, the theory of principal bundles with connections provides a natural setting. The computation of geometric phases is perhaps best understood in this setting.

In previous work [5] [6],, we have given explicit formulas for geometric phases in various examples involving coupled rigid and flexible bodies. The abstract framework for such formulas is based on the following ingredients:

(a) a simple mechanical system with symmetry $(Q, K, V, G)$ with configuration space $Q$, kinetic energy quadratic form (or riemannian metric on $Q$) denoted by $K$, a Lie group $G$ acting on $Q$ leaving invariant $K$ and the potential energy $V$;

(b) a principal $G$-bundle $(Q, Q/G, G)$ where the space $S = Q/G$ is known as the shape space;

(c) controls (forces) acting on $(Q, K, V, G)$, also leaving invariant the conserved momentum map $J^\sharp : TQ \to \mathcal{G}^*$ the dual of the Lie algebra of $G$, associated to the free hamiltonian system with energy $K + V$.

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The map $J^\sharp$ is given explicitly by the formula

$$J^\sharp(w_q)\xi = (K^\flat w_q)(\xi_Q(q)) \tag{1.1}$$

where $\xi \in \mathcal{G}$ the Lie algebra of $G$, $K^\flat$ is the Legendre transform and $\xi_Q$ is the infinitesimal generator (vector field on $Q$) associated to $\xi$. Let $\mathbf{I}_q$ denote the symmetric bilinear form on $\mathcal{G}$,

$$\mathbf{I}_q(\xi, \eta) = K(\xi_Q(q), \eta_Q(q)). \tag{1.2}$$

Let $\mathbf{I}_q^\flat : \mathcal{G} \to \mathcal{G}^*$ be the corresponding pairing. Then we have a vertical-horizontal splitting of the tangent bundle $TQ$,

$$\begin{aligned} TQ_q &= (Vert)_q \oplus (Hor)_q \\ w_q &= ((\mathbf{I}_q^\flat)^{-1}\mu)_Q(q) + (w_q - ((\mathbf{I}_q^\flat)^{-1}\mu)_Q(q)) \end{aligned} \tag{1.3}$$

where $\mu = J^\sharp(w_q)$. This splitting has the equivariance property with respect to the $G$-action on $Q$ and defines a principal connection. This *mechanical connection* appears to be originally due to Smale and Kummer (see [7]).

In the concrete setting of *planar* many-body problems with rigid and flexible components, the geometric phase shift (angle) associated to a specific path $\gamma$ in shape space $S$ is simply the integral of the connection form over the path. For loops $\gamma$ it is the *holonomy*. Using the formulas for geometric phases, it is possible to derive motion plans with prescribed holonomy. In the forthcoming Ph.D. thesis of R.Yang [9], formulas for the curvature of the mechanical connection are exploited for this purpose. See also Byrne's MS thesis [3].

Given the above framework, there is a natural class of associated kinematic control problems of the form

$$\dot{q} = X_\mu(q) + D(q)u, \tag{1.4}$$

where $X_\mu(q) = ((\mathbf{I}_q^\flat)^{-1}\mu)_Q(q)$ is the drift, $u \in T_{\pi(q)}(Q/G)$ is a tangent vector on shape space representing controls, and $u \mapsto D(q)u$ is the horizontal lift. Consider the following control problems:

(P1) Given $q^0$ and $q^1$ find $u(\cdot)$ steering $q^0$ to $q^1$ at a specified time;

(P2) Given $q^0$ and $q^1$ in the same $G$-orbit as $q^0$ find $u(\cdot)$ steering $q^0$ to $q^1$ while minimizing

$$\int_0^T < u, u >_S dt$$

for a given riemannian metric $< \cdot, \cdot >_S$ on $S = Q/G$.

When $\mu = 0$, i.e. in the driftless case, various methods are known for the constructive controllability problem (P1) (due to Hermes, Laferriere and Sussman, Sastry and Murray, etc).

The variational problem (P2) has been studied by Montgomery who refers to it as the *isoholonomy problem* when

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$\mu = 0$. Montgomery gave necessary conditions for optimal paths. In the case when the metric on $S$ is the induced metric, the differential equations satisfied by optimal paths are the equations of motion for a particle in a Yang-Mills field. Of course from the point-of-view of optimal control, these are also simply the hamilton equations from Pontryagin's maximum principle. The symmetry group $G$ holds for these equations. Special instances of such variational problems appear in the Ph.D. dissertation of J. Baillieul [1].

For the planar N-body problem with flat metric on shape space $T^{N-1}$ considered in [5] and [6], the hamilton equations of interest are of the form:

$$\begin{aligned} \dot{q} &= D(q)D^T(q)p \\ \dot{p} &= -\frac{\partial}{\partial q}\left(\frac{1}{2}p^T D(q)D^T(q)p\right). \end{aligned} \tag{1.5}$$

Quite independent of the mechanical motivations mentioned earlier, Vershik and Gershkovich have investigated a class of variational problems leading to a type of geometry that they refer to as nonholonomic geometry (see [4] and references therein). Some of the same circle of ideas were investigated earlier by Brockett [2].

The variational problems on principal bundles as outlined above are quite difficult to solve. There are two approaches to simplification. One is symplectic reduction of the hamilton equations by the symmetry group (structure group of the principal bundle). One can hope to obtain tractable reduced problems. The alternative is to consider localization of the optimal control problem (P2) about a reference *shape*. By a localization we mean a local normal form. It has been known for sometime that there exist explicitly solvable localizations. We note the "area-rule" normal form due to Brockett [2]:

$$\min \int_0^1 (u_1^2 + u_2^2)dt$$

$$\begin{aligned} \dot{x} &= u_1 \\ \dot{y} &= u_2 \\ \dot{z} &= xu_2 - yu_1 \end{aligned} \tag{1.6}$$

and boundary conditions $x(0) = x(1), y(0) = y(1)$ and $z(0), z(1)$ specified. As Brockett notes in [2], the difference $z(1) - z(0)$ is the area of a Lissajous figure and the optimal controls are sinusoids.

In contrast the "area-moment" normal form of [6]:

$$\min \int_0^1 (u_1^2 + u_2^2)dt$$

subject to

$$\begin{aligned} \dot{x} &= u_1 \\ \dot{y} &= u_2 \\ \dot{z} &= x^2u_2 - y^2u_1 \end{aligned} \tag{1.7}$$

for given boundary conditions $x(0) = x(1), y(0) = y(1)$ and $z(0), z(1)$ specified, is explicitly solvable by *elliptic functions*.

To see this, note that the differential equations for geodesics in this case are:

$$\begin{aligned} \ddot{x} - \lambda(x+y)\dot{y} &= 0 \\ \ddot{y} + \lambda(x+y)\dot{x} &= 0 \\ \ddot{z} + 2(y-x)\dot{x}\dot{y} + \lambda(x+y)(x^2\dot{x} + y^2\dot{y}) &= 0. \end{aligned} \tag{1.8}$$

There is a first integral $\dot{x}^2 + \dot{y}^2$. Here $\lambda$ is a Lagrange multiplier. Letting $w = x + y$, it is easy to see that, for a constant $c$,

$$\ddot{w} + \lambda w\left(c + \frac{\lambda}{2}w^2\right) = 0, \tag{1.9}$$

the equation for a quartic oscillator, solvable by elliptic functions.

The area-moment normal form appears concretely as a localization of a model mechanical variational problem involving a planar rigid body and two point masses[6]. The underlying symmetry group is $S^1$. The bundle is $(R^2 \times S^1, R^2, S^1)$. The mechanical connection 1-form is approximately,

$$\omega = \frac{m\epsilon q}{I + 2m\epsilon q^2}(x^2dy - y^2dx), \tag{1.10}$$

where $m, \epsilon, q, I$ are mechanical constants. It is an easy step to proceed from this approximation to the area-moment normal form. A non-abelian analog of this mechanical example with $SO(3)$ as the symmetry group is obtained by considering a three dimensional rigid body with two driven point masses constrained to move along linear paths. In this case again, the bundle is topologically trivial. It is $(R^2 \times SO(3), R^2, SO(3))$. However the mechanical connection is nontrivial and the holonomy Lie algebra is all of $so(3)$. It follows that there exists a motion plan (i.e. solution to (P1)) for the driving masses that steers the rigid body from any initial orientation to an arbitrary new orientation.

The optimal control problem (P2) in this case has interesting geometric structure. It admits $SO(3)$ as a symmetry group. In section 2 we set up the model nonabelian problem of a rigid body with two oscillators and discuss the associated connection. In section 3 we state results on controllability and give the necessary conditions for optimal control (reorientation). Section 4 is about simplification via Poisson reduction.

## 2. Rigid Body with Oscillators

Figure 1 represents a mechanical system consisting of a three dimensional rigid body carrying two point masses that are confined to move along guide ways. The point masses are to be viewed as driven oscillators (not necessarily harmonic in character).

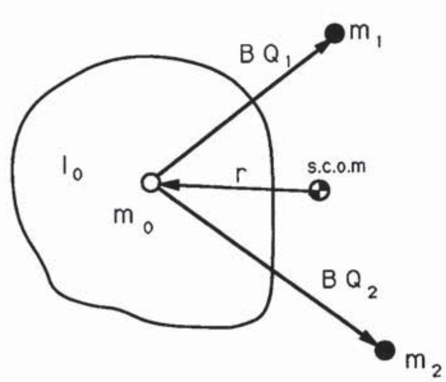


Fig. 1.

Here, **r** is the vector from the center of mass of the system to the center of mass of the carrier body, $B \in SO(3)$ represents the orientation of the carrier with respect to the inertial frame,

$Q_1$ and $Q_2$ are the position vectors of the oscillators with respect to the frame fixed on carrier. Assume to start with that $Q_1$ and $Q_2$ are arbitrary time dependent vectors. Later on, we will impose constraints on them to study the effect of their motion on the motion of the carrier. The angular momentum, $\mu$, of the system can be obtained by direct calculation as

$$\mu = BI_0\Omega + m_0 r \times \dot{r} + \sum_{i=1}^{2} m_i(BQ_i + r) \times (\dot{B}Q_i + B\dot{Q}_i + \dot{r}) \quad (2.1)$$

where $\Omega$ is the body angular velocity of the carrier in the body frame fixed to the carrier. Note that $I_0$ is the moment of inertia of the carrier and,

$$r = -B(\epsilon_1 Q_1 + \epsilon_2 Q_2)$$

where $\epsilon_i = \frac{m_i}{m}$, $i = 1,2$ and $m = m_0 + m_1 + m_2$, and that

$$\dot{B} = B\widehat{\Omega}$$

where the standard isomorphism

$$\hat{\ }: R^3 \to so(3)$$

$$(x_1, x_2, x_3) \mapsto \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

holds. We then get

$$\mu = B((I_0 + \Delta I_0)\Omega + D_1\dot{Q}_1 + D_2\dot{Q}_2) \quad (2.2)$$

where

$$\Delta I_0 = -m(\epsilon_1\widehat{Q}_1^2 + \epsilon_2\widehat{Q}_2^2 - (\epsilon_1\widehat{Q}_1 + \epsilon_2\widehat{Q}_2)^2)$$
$$D_1 = m[(1-\epsilon_1)\epsilon_1\widehat{Q}_1 - \epsilon_1\epsilon_2\widehat{Q}_2]$$
$$D_2 = m[-\epsilon_1\epsilon_2\widehat{Q}_1 + \epsilon_2(1-\epsilon_2)\widehat{Q}_2].$$

As outlined in the Introduction, the principal bundle $(R^2 \times SO(3), R^2, SO(3))$ associated to the present setting admits a principal connection. The horizontal distribution is the kernel of an $so(3)$ valued connection 1-form given by

$$\begin{aligned}\hat{\omega} = -mI_{lock}^{-1}[((1-\epsilon_1)\epsilon_1\widehat{Q}_1 - \epsilon_1\epsilon_2\widehat{Q}_2)dQ_1 \\ + (-\epsilon_1\epsilon_2\widehat{Q}_1 + \epsilon_2(1-\epsilon_2)\widehat{Q}_2)dQ_2]\end{aligned} \quad (2.3)$$

where $I_{lock} = I_0 + \Delta I_0$. Formula (2.3) has another interpretation. Assume that $\mu = 0$. If the point masses are driven to move at rates $\dot{Q}_1$ and $\dot{Q}_2$, the carrier body will undergo a drift given by the body angular rate,

$$\begin{aligned}\Omega = -mI_{lock}^{-1}[((1-\epsilon_1)\epsilon_1\widehat{Q}_1 - \epsilon_1\epsilon_2\widehat{Q}_2)\dot{Q}_1 \\ + (-\epsilon_1\epsilon_2\widehat{Q}_1 + \epsilon_2(1-\epsilon_2)\widehat{Q}_2)\dot{Q}_2]\end{aligned} \quad (2.4)$$

This is the phenomenon of pure geometric phase drift in a 3 dimensional setting.

Now assume that $m_1 = m_2$. On the carrier, a coordinate system $(0\text{-}xyz)$ is set such that the moment of inertia of the rigid body satisfies $I_0 = diag(I_x, I_y, I_z)$ i.e. $0\text{-}x$, $0\text{-}y$, $0\text{-}z$ axes are principal axes. Let $Q_1$ and $Q_2$ be given by

$$Q_1(t) = Q_0 + p_1 x_1(t), \qquad Q_2(t) = -Q_0 + p_2 x_2(t)$$

where $Q_0 = (0,0,q)^T$, $p_i = (cos(\psi_i), sin(\psi_i), 0)^T$ and $x_i(t) \in R, \forall t$ with constants $q$ and $\psi_i$ for $i = 1,2$. This means that the two point masses are restricted to move along lines which are parallel to the $0\text{-}xy$ plane and are at equal distance from the plane (see Fig. 2.).

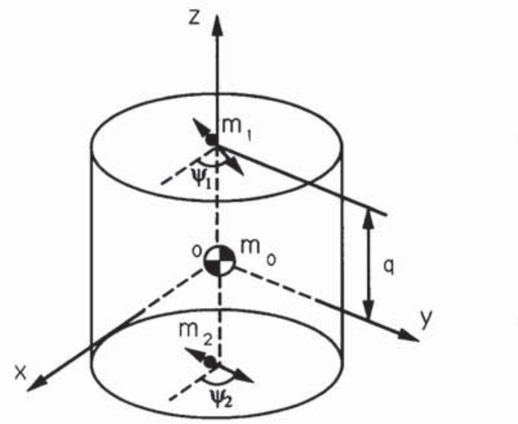


Fig. 2.

With above setting, the connection form (2.3) can be re-expressed as

$$\hat{\omega} = \hat{\Omega}_1(x_1,x_2)dx_1 + \hat{\Omega}_2(x_1,x_2)dx_2, \quad (2.5)$$

where $\Omega_1$ and $\Omega_2$ can be further expressed as

$$\Omega_1 = \frac{1}{det(I_{lock})}\begin{pmatrix}\omega_{11}\\ \omega_{12}\\ \omega_{13}\end{pmatrix} \qquad \Omega_2 = \frac{1}{det(I_{lock})}\begin{pmatrix}\omega_{21}\\ \omega_{22}\\ \omega_{23}\end{pmatrix}$$

where $det(I_{lock})$ and $\omega_{ij}$ for $i = 1,2; j = 1,2,3$ are polynomials of $x_1$ and $x_2$. For reasons of space, we omit the long formulas for $det(I_{lock})$ and $\omega_{11}$ etc..

Special choices of the $\psi_i$ lead to simplifications. Suppose $\psi := \psi_1 = \psi_2$. Then

$$\omega_{13} = -\omega_{23} = \frac{1}{2}\epsilon^2(I_y - I_x)m^2q^2 sin(2\psi)(x_2 - x_1).$$

Thus if $I_x = I_y$ or if $\psi = 0$ or $\pi/2$, then $\omega_{13} = \omega_{23} = 0$, i.e. the rigid body will only drift (rotate) about the axis perpendicular to the plane formed by the tracks of two point masses. In general, the parallel motion of the two point masses will cause the rigid body to drift about the $z$ axis. In other words, when $\psi_1 = \psi_2$ holds, a sufficient condition for planar drift is that the carrier has axial symmetry about $z$-axis, or that two point masses move along the lines which are parallel with the same principal axis ($0\text{-}x$ or $0\text{-}y$) of the carrier.

Explicitly, if $\psi = 0$, i.e. both point masses move parallel to principal axis $0\text{-}x$,

$$\omega = \begin{pmatrix} 0 \\ \frac{\epsilon mq(dx_1 - dx_2)}{m\epsilon(\epsilon-1)x_2^2 + 2\epsilon^2 m x_1 x_2 + \epsilon m(\epsilon-1)x_1^2 - 2\epsilon m q^2 - I_y} \\ 0 \end{pmatrix}.$$

If $\psi = \frac{\pi}{2}$, i.e. both point masses move parallel to principal axis $0\text{-}y$,

$$\omega = \begin{pmatrix} -\frac{\epsilon mq(dx_1 - dx_2)}{m\epsilon(\epsilon-1)x_2^2 + 2\epsilon^2 m x_1 x_2 + \epsilon m(\epsilon-1)x_1^2 - 2\epsilon m q^2 - I_x} \\ 0 \\ 0 \end{pmatrix}.$$

These formulas for $\omega$ can be shown to agree after a Taylor series truncation to the formula (1.10) associated with planar localization.

**Remark:** The model problem described above is strongly motivated by a troublesome phenomenon of drift observed in the Hubble Space Telescope due to thermo-elastically driven vibrations of the solar panels arising from the day-night thermal cycling on-orbit. The point mass oscillators in our problem may be viewed as one-mode truncations of the elasto-mechanical problem.

### 3. Optimal Reorientation

From the discussion of the previous section, it should be apparent that if one is interested in full three dimensional motion of the carrier body, some skewness in the directions of particle motions would be necessary. Here we set $\psi_1 = 0$ and $\psi_2 = \frac{\pi}{2}$, and study the optimal control problem for this setting.

Now the constraint on the motion of two point masses becomes

$$Q_1 = -qe_3 + x_1(t)e_1 \qquad Q_2 = qe_3 + x_2(t)e_2$$

where $e_i$ for $i = 1, 2, 3$ are the standard unit vectors in $R^3$. The body angular velocity $\Omega$ is given by (2.4).

Now, the optimal control problem is to find $u_1(\cdot)$ and $u_2(\cdot)$ to

$$\text{minimize} \int_0^1 u_1^2 + u_2^2 dt \tag{3.1a}$$

such that

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{A} &= A(\widehat{\Omega}_1 u_1 + \widehat{\Omega}_2 u_2) \end{aligned} \tag{3.1b}$$

for given boundary conditions

$$x_1(0) = x_1(1), \qquad x_2(0) = x_2(1)$$

$$A(0) = A_0 \in SO(3), \qquad A(1) = A_1 \in SO(3) \tag{3.1c}$$

where $A(t) \in SO(3)$ for any $t \in [0, 1]$ represents the orientation of the carrier in inertial space.

Let $p = (x, A)$, for $x = (x_1, x_2)$. Eq(3.1b) can be represented as

$$\dot{p} = X_1(p)u_1 + X_2(p)u_2 \tag{3.1b$'$}$$

where

$$X_1(p) = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, A\widehat{\Omega}_1(x))$$

and

$$X_2(p) = (\begin{pmatrix} 0 \\ 1 \end{pmatrix}, A\widehat{\Omega}_2(x)).$$

The vector fields $X_i$ agree with the general form of left invariant vector fields

$$X_i(x, A) = (F_i(x), A\widehat{G}_i(x)) \tag{3.2}$$

on bundle $(R^2 \times SO(3), R^2, SO(3))$.

**Proposition 3.1**: Given two left invariant vector fields $X_1$ and $X_2$ on the principal bundle $(R^2 \times SO(3), R^2, SO(3))$, the Lie bracket of $X_1, X_2$ is given by

$$\begin{aligned} &[X_1, X_2](x, A) \\ &= (\frac{\partial F_2}{\partial x}F_1 - \frac{\partial F_1}{\partial x}F_2, \;\; A(\widehat{G_1 \times G_2 + \frac{\partial G_2}{\partial x}F_1 - \frac{\partial G_1}{\partial x}F_2})\,) \end{aligned} \tag{3.3}$$

for any point $(x, A) \in R^2 \times SO(3)$. Here notation $\widehat{(\cdot)} \equiv \widehat{(\cdot)}$. ∎

The system (3.1b) is in the category of driftless systems. Using the Lie bracket formula (3.3), the well-known Lie algebra rank condition, and Chow's theorem, it is possible to state a controllability theorem.

**Theorem 3.2**: For the system (3.1), there is an open set about $(0, 0, A)$ such that any point in this set can be reached from $(0, 0, A)$ by a piecewise constant input $(u_1, u_2)$.

*Proof:* It is a long complicated calculation using Macsyma and is omitted. ∎

The problem of determining the geometric equations can be approximated in several ways. As mentioned in section 1, Montgomery [8] gives general arguments to show that the geodesic equations are equations of motion of a particle in Yang-Mills field (determined by the mechanical connection). One can appeal to the maximum principle and get the same equations. Considerable care is needed to take proper account of the geometry of $T(R^2 \times SO(3))$. Omitting the details we state,

**Theorem 3.3** If $(x(\cdot), A(\cdot))$ is an optimal trajectory with controls $(u_1^*(\cdot), u_2^*(\cdot))$ for the optimal control problem given by Eq. (3.1), then there exist $\mu(\cdot) = (\mu_1(\cdot), \mu_2(\cdot))$ and $\lambda(\cdot)$ on $[0, 1]$ satisfying the ordinary differential equations with boundary conditions:

$$\begin{aligned} \dot{x}_1 &= u_1^* \\ \dot{x}_2 &= u_2^* \\ \dot{A} &= A(\widehat{\Omega}_1 u_1^* + \widehat{\Omega}_2 u_2^*) \\ \dot{\mu}_1 &= -\lambda^T(\frac{\partial \Omega_1}{\partial x_1}u_1^* + \frac{\partial \Omega_2}{\partial x_1}u_2^*) \\ \dot{\mu}_2 &= -\lambda^T(\frac{\partial \Omega_1}{\partial x_2}u_1^* + \frac{\partial \Omega_2}{\partial x_2}u_2^*) \\ \dot{\lambda} &= \lambda \times (\Omega_1 u_1^* + \Omega_2 u_2^*) \end{aligned} \tag{3.4}$$

where

$$u_1^*(x, A, \mu, \lambda) = -\frac{1}{2}(\mu_1 + \lambda^T \Omega_1)$$

$$u_2^*(x, A, \mu, \lambda) = -\frac{1}{2}(\mu_2 + \lambda^T \Omega_2)$$

and $x_1(0) = x_1(1) = x_2(0) = x_2(1) = 0$ and $A(0) = A_0 \in SO(3), A(1) = A_1 \in SO(3)$. ∎

**Remark 3.1**: For each $t$, the vector $\lambda(t) \in R^3$ should be correctly viewed as an element in the dual of the Lie algebra of $SO(3)$. $\mu_1$ and $\mu_2$ are scalar co-state variables associated to the base space of the principal bundle.

**Remark 3.2**: Eq.(3.4) defines a hamiltonian vector field on the symplectic manifold $T^*(R^2 \times SO(3))$, once the substitutions are made for $u_1^*$ and $u_2^*$. It is this vector field that we investigate further in Section 4, from the viewpoint of explicit integrability of (3.4).

### 4. Symmetry and Reduction

We have as yet made no simplifications or approximations of the optimal control problem. It is customary to make ad hoc approximations in the interest of ensuring analytic integrability or numerical solvability. However, in the process one can easily destroy symmetries inherent to the problem. This is highly undesirable.

In the present section we indicate how to take account of symmetries, do reduction to a lower dimensional problem and then do approximations. Such a reduction may even be done in stages.

Recall that the manifold $P = T^*(R^2 \times SO(3))$, parameterized by $z = (x, A, \mu, A\hat{\lambda})$, is symplectic, and hence Poisson. The hamiltonian constructed from the optimal control problem is

$$H(x, A, \mu, A\hat{\lambda}) = -\frac{1}{4}\sum_{i=1}^{2}(< \mu, f_i > + < \lambda, \Omega_i >)^2, \tag{4.1}$$

where $f_1 = (1,0)^T, f_2 = (0,1)^T$. The corresponding hamiltonian vector field $X_H$ is determined easily with respect to the canonical symplectic structure on $P$. The integral curves of $X_H$ satisfy the differential equations (3.4).

Consider the action of $SO(3)$ on $R^2 \times SO(3)$

$$\Phi : SO(3) \times (R^2 \times SO(3)) \to R^2 \times SO(3)$$
$$(B, (x, A)) \mapsto (x, BA)$$

and its cotangent lift on $T^*(R^2 \times SO(3))$

$$\Phi^{T^*} : SO(3) \times T^*(R^2 \times SO(3)) \to T^*(R^2 \times SO(3))$$
$$(B, (x, A, \mu, A\hat{\lambda})) \mapsto (x, BA, \mu, A\hat{\lambda})$$

The quotient space $T^*(R^2 \times SO(3))/SO(3) \simeq R^2 \times R^2 \times R^3$ coordinatized by $(x, \mu, \lambda)$. It is obvious that the hamiltonian (4.1) is invariant under the action $\Phi^{T^*}$ since it does not have $A$ in its expression. Hence the reduced hamiltonian $\tilde{H}$ on $T^*(R^2 \times SO(3))/SO(3)$ is simply

$$\tilde{H}(x, \mu, \lambda) = H(x, A, \mu, A\hat{\lambda}) \tag{4.2}$$

where $\tilde{H}$ is defined as

$$\tilde{H} \circ \pi_1 = H$$

for canonical projection $\pi_1 : T^*(R^2 \times SO(3)) \to T^*(R^2 \times SO(3))/SO(3)$. Because of (4.2) we will not distinguish between $H$ and $\tilde{H}$ later. The corresponding reduced vector field $X_H$ is simply given by (3.4) without the third equation,. i.e.

$$\begin{aligned}
\dot{x}_1 &= u_1^* \\
\dot{x}_2 &= u_2^* \\
\dot{\mu}_1 &= -\lambda^T\left(\frac{\partial \Omega_1}{\partial x_1} u_1^* + \frac{\partial \Omega_2}{\partial x_1} u_2^*\right) \\
\dot{\mu}_2 &= -\lambda^T\left(\frac{\partial \Omega_1}{\partial x_2} u_1^* + \frac{\partial \Omega_2}{\partial x_2} u_2^*\right) \\
\dot{\lambda} &= \lambda \times (\Omega_1 u_1^* + \Omega_2 u_2^*)
\end{aligned} \tag{4.3}$$

where

$$u_1^*(z) = -\frac{1}{2}(\mu_1 + \lambda^T \Omega_1)$$
$$u_2^*(z) = -\frac{1}{2}(\mu_2 + \lambda^T \Omega_2).$$

The above reduction procedure is in fact Poisson reduction. Recall that given a group $G$, $(P, \{\ ,\ \})$ is Poisson reducible if $P/G$ has a Poisson structure $\{\ ,\ \}_{P/G}$ such that for any smooth functions $f, h \in C^\infty(P/G)$ and smooth functions $F$ and $H \in C^\infty(P)$ given by $f \circ \tau$ and $h \circ \tau$, respectively, we have

$$\{F, H\}_P = \{f, h\}_{P/G} \circ \tau$$

where $\tau$ is projection $P \to P/G$. The reduced Poisson structure for our problem is given by

$$\{f, h\}_{P/SO(3)}(x, \mu, \lambda) = (\nabla f)^T \Lambda \nabla h, \tag{4.4}$$

where

$$\Lambda(z) = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & \hat{\lambda} \end{pmatrix} \tag{4.5}$$

where $I$ is the $2 \times 2$ identity matrix. $\Lambda$ is the reduced Poisson tensor. It is immediate that for the reduced system (4.3) which agrees with the reduced vector field $\Lambda(z)\nabla H(z)$, the function $\|\lambda\|^2$ is a conserved quantity (Casimir function).

It is obvious that from the solutions of the reduced equations one can determine those of the unreduced system by quadrature. In the present setting the reduced equations are not any more tractable than the original ones. Further work is necessary. The idea is to

(a) approximate $H$ (e.g. by Taylor polynomials);
(b) introduce additional symmetries;
(c) reduce further;
(d) check integrability.

In the present setting we approximate $H$ by

$$H_{12} = -\frac{1}{4}((b\lambda_3 x_2 + \mu_1 + a\lambda_2)^2 + (b\lambda_3 x_1 - \mu_2 - a\lambda_1)^2) \tag{4.6}$$

This is justified by Taylor series truncation and *the additional symmetry condition $I_z = I_y$ inspired by the symmetric heavy top*. The corresponding hamiltonian vector field $X_{12}$ is given by $\Lambda \nabla H_{12}$.

Consider the linear transformation group action on the Poisson reduced space $P/SO(3) \simeq R^7$ given by

$$\Psi : S^1 \times R^7 \to R^7$$
$$(\tau, (x, \mu, \lambda)) \mapsto (Rot(\tau)x, Rot(\tau)\mu, Rot^3(-\tau)\lambda)$$

where $Rot(\tau)$ denotes the usual rotation by angle $\tau$ in the plane and $Rot^3(-\tau)\lambda$ denotes rotation by angle $(-\tau)$ in the $\lambda_1, \lambda_2$ plane, leaving $\lambda_3$ fixed. We note the following striking fact,

**Proposition 4.1**: The action $\Psi$ of $S^1$ leaves invariant the reduced Poisson structure *and* the approximate hamiltonian $H_{12}$.

*Proof*: The proof is a long calculation. ■

At this stage, taking into account the first integrals in each reduction, we can show that the hamiltonian equations for the optimal control problem can be reduced to a 4-dimensional systems. The explicit integrability of those systems is of interest. For suitable constants one can expect integrability. Further details will appear in a forthcoming paper.

## 5. Conclusions

In this paper we have discussed a generalization of the rigid body problem to a (degenerate) optimal control problem on an $SO(3)$-bundle. The methods of reduction are used to address simplification and integrability.

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