GEOMETRY OF PARAMETRIC MODELS: SOME PROBABILISTIC QUESTIONS

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ABSTRACT

Input-output maps with parameters appear naturally as a consequence of certain scalings. We discuss here the role of these scalings in relation to the geometry of rational functions. These geometric questions appear to be of interest for identification problems. In this connection we pose a problem of setting up densities on the space of models and examine some solutions. Our investigations make interesting contact with analytical mechanics.

INTRODUCTION

A theory of scaling for rational functions first introduced in [1] leads naturally to the notion of i/O maps with parameters. The first part of this paper is a collection of results on the geometry of input-output maps. We then investigate the implications of these results for identification experiments and propose a Bayesian framework for statistical inference on systems. In the second part we examine some invariants of scaling and show how these lead to densities on transfer functions.

1. PARAMETRIC MODELS

Dynamical systems with parameters have been studied extensively in connection with questions on qualitative behavior of differential equations [2]. In the context of input-output systems, transfer functions with parameters appear as a consequence of an experimenter's choice of optimum/convenient conditions. To illustrate, associated with a $g(s) \epsilon$ Rat(n), the set of proper rational functions of McMillan degree n without common factors we have the four-parameter family

$$F = \left\{ \frac{mg(\alpha s+\sigma)}{1+mk\ g(\alpha s+\sigma)} : \alpha \epsilon R^+,\ m \epsilon R^+,\ \sigma \epsilon R,\ k \epsilon R \right\}$$

Here, m and α are magnitude and frequency scalings and k is a feedback gain. The parameter σ effects exponential scaling. We make precise these ideas by defining G-scaling of a smooth manifold M by a Lie group G to be a smooth G-action, $\phi : G \times M \to M$. Points in the same orbit are then thought of as scaled versions of each other. In the present context we are interested in scalings of Rat(n) carrying a smooth manifold structure [3]. Thus, the four-parameter family F of transfer functions defined above is a typical orbit of the action of the group G_A generated by the four one-parameter scalings:

(a) $s \to \alpha s \quad ; \quad \alpha \epsilon R^+$

(b) $s \to s + \sigma \quad ; \quad \sigma \epsilon R$

(c) $g(s) \to mg(s) \quad ; \quad m \epsilon R^+$

(d) $g(s) \to \dfrac{g(s)}{1 + kg(s)} \quad ; \quad k \epsilon R$

It was indicated in [1] (see [4] for proof) that G_A is isomorphic to the group of 4 x 4 matrices of the form:

$$\begin{bmatrix} 1 & \sigma & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & mk \\ 0 & 0 & 0 & m \end{bmatrix}$$

By introducing a fifth scaling, time shift,

$$\text{(e)} \quad g(s) = c(sI-A)^{-1}b \to c(sI-A)^{-1} e^{A\tau}b \qquad \tau \in R$$

(or equivalently)

$$ce^{At}b = w(t) \to w(t+\tau) ,$$

and combining this with the scalings (a), (b) and (c) above, we have another group G_B acting on Rat(n). We recall from [1] that G_B has a matrix representation.

$$\begin{bmatrix} 1 & \tau & \ell n(m) \\ 0 & \alpha & \sigma \\ 0 & 0 & 1 \end{bmatrix}$$

We are interested in the disposition of the orbits and their structure. For example, one may ask if the orbits are connected and whether all orbits have the same dimension as integral submanifolds of Rat(n). First we need a lemma.

Lemma 1: The scaling operations (a)-(e), preserve the McMillan degree. Further, all except (e) leave the number of zeros unchanged.

The proof is easy (see [4]). The requirement that there be no common factors disconnects the space Rat(n).

One of the main results in [3] was the following.

Thm 1: (Brockett) Rat(n) has (n+1) arcwise connected components. All members of a particular component have the same Cauchy index which takes values in the set $\{-n, -n+2, \ldots, n-2, n\}$.

Denoting as Rat(p,q) the connected component of Rat(p+q) of Cauchy index (p-q), we see that Lemma 1 implies that orbits of G_A and G_B lie entirely within some Rat(p,q). Further, since the groups G_A and G_B are connected Lie groups, each orbit is connected. Now, for every smooth action ϕ by a group G on a smooth manifold M we have the following: To every $m \in M$, there corresponds a C^∞ map denoted by m also of G into M defined by

$$m(g) = gm = \phi(g,m)$$

Let dm denote the corresponding derivative map from the tangent space at e (Lie algebra $\tilde{G}$) of G. Then we have the following Lie algebra homomorphism,

$$\lambda: \tilde{G} \to U(M) = \text{Lie algebra of } C^\infty \text{ vector fields on } M$$

$$X \to \overline{X} = \lambda X \text{ defined by } (\lambda X)\ (m) = dm(X(e))$$

The orbits of G are integral submanifolds of M of constant dimension if dm is kernel-free for all $m \in M$. This follows from the Frobenius theorem [5]. The condition on dm is the same as saying that G acts freely on M. This question was first raised in connection with G_A action on Rat(n) in [1] and there a preliminary condition was given. Complete results and proofs are in [4] and will appear elsewhere. We state,

Thm 2: The groups G_A and G_B act freely (hence effectively) on Rat(p,q) iff $|p-q| > 1$, in which case Rat(p,q) can be given the structure of a fiber bundle with structure group G_A or G_B.

The situation is complicated by the fact that the natural map $\Pi: \text{Rat}(p,q) \to \text{Rat}(p,q)/G$ for $G = G_A$ or G_B need not be a fiber map [6,7]. More precisely Rat(p,q) is a (trivial) principal G_A-bundle over Q_A

$$\mathcal{B}_A \triangleq (\text{Rat}(p,q), \tilde{\Pi}_A, Q_A, G_A)$$

$$\text{Rat}(p,q) \tilde{=} Q_A \times G_A$$

where the base space Q_A is the set of rational functions of the form

$$g(s) = \frac{s^{m-1} + 0 \cdot s^{m-2} + q_{m-3}s^{m-3} + \ldots + q_{\ell+1}s^{\ell+1} + 1\ s^{\ell}}{s^n + p_{n-1}s^{n-1} + \ldots + p_1 s + p_0}$$

with $(q_{n-1}, q_{n-2}, \ldots, q_0)$ orthogonal to $(p_{n-1}, p_{n-2}, \ldots, p_0)$. A similar smooth orbit-canonical form for G_B exists (see [4] for details). At the level of realizations also, we can exhibit such structure. Denoting as $\hat{R}$ the set of all minimal triples [A, b, c]. We have a trivial bundle with structure group (fiber) $G\ell(n)$,

$$(\hat{R}, \Pi, \text{Rat}(n), G\ell(n))$$

$$\hat{R} \cong \text{Rat}(n) \times G\ell(n)$$

with

$$\Pi([A, b, c]) = c(sI-A)^{-1}b.$$

We see that parametric sets of i/o maps can be given a precise geometric structure under some conditions. Since the scalings of this section are dependent on an experimenter's free choice, there appears to be an element of ambiguity with regard to the outcome of identification experiments and tests of hypotheses about systems. We examine this and related questions below.

2. IMPLICATIONS FOR SYSTEM IDENTIFICATION

To focus ideas, consider the concrete problem of identifying the system,

$$(*) \quad \begin{aligned} x_{t+1} &= Ax_t + bu_t \\ y_t &= cx_t \end{aligned}$$

where $x_t \in R^n$, $A \in R^{n\times n}$, $b \in R^{n\times 1}$, $c \in R^{1\times n}$ and $\{u_t\}$ is discrete-time standard Gaussian white noise. To identify the transfer function $g(z) = c(zI-A)^{-1}b$, let $I(\theta, \hat{\theta}, y, T)$ be the support function or identification criterion, i.e. if θ represents the true transfer function and θ an estimate then $I(\theta, \hat{\theta}, y, T)$ is the measure of support for $\hat{\theta}$, given by a realization $y(\cdot)$ of length T. A commonly used support function is the log-likelihood function. To eliminate ambiguity due to scalings, one asks for invariance in the sense of [1], i.e.

$$I(\theta, \hat{\theta}, y, T) = I(g(\theta), g(\hat{\theta}), g(y), g(T)) \text{ wp1} \quad \forall g \in G$$

where G is a (scaling) Lie group acting on Rat(n).

This question was examined in detail in [1] in relation to (*), using the asymptotic expression of support,

$$V(\theta,\hat{\theta}) = \frac{1}{2\pi i} \oint_C \frac{g(z)g(z^{-1})}{\hat{g}(z)\,\hat{g}(z^{-1})} \frac{dz}{z}$$

where C is the circle, $|z| = 1$. Let us denote the transfer function as

$$g(z) = \frac{1 + q_{m-1} z^{-1} + \dots + q_0 z^{-m}}{1 + p_{n-1} z^{-1} + \dots + p_0 z^{-n}}$$

The question of main interest to us is the effect of the geometry of Rat(n) on the identification criterion and V is a convenient object for this study.

Firstly, V is insensitive to common factors. Secondly, in a variety of practical problems, it is hard to arrive at a proper guess of the bidegree (n,m) where n = number of poles and m = number of zeros. As a result, some trial and error or more formally hypothesis-testing is involved. Since an estimate $\hat{g}(z)$ that achieves the maximum of log-likelihood or equivalently minimum of V is often desirable [8], it is necessary to understand the geometry of the set of minima of $V = V(\hat{q},\hat{p})$ as a function of

$$\hat{g}(z) = \frac{\hat{q}(z)}{\hat{p}(z)} = \frac{1 + \hat{q}_{\hat{m}-1} z^{-1} + \dots + \hat{q}_0 z^{-\hat{m}}}{1 + \hat{p}_{\hat{n}-1} z^{-1} + \dots + \hat{p}_0 z^{-\hat{n}}}$$

We shall summarize below some results due to Åström and Söderström [9]. First some definitions.

Def^{n}1: Let $\mathrm{DRat}_{SM}(n;m) \triangleq$ set of all rational functions without common factors with McMillan degree = n and number of zeros = $m \leq n$ and all poles and zeros lying in the open unit disk.

Def^{n}2: Let $\overline{\mathrm{DRat}}_{SM}(n;m) \supset \mathrm{DRat}_{SM}(n;m)$ be the set obtained by removing the restriction on common factors in Definition 1. The following is a version of the reults in [9].

Thm 3: (Åström and Söderström): Let

$$g(z) = \frac{1 + q_{m-1} z^{-1} + \dots + q_0 z^{-m}}{1 + p_{n-1} z^{-1} + \dots + p_0 z^{-n}}$$

$q_0 \neq 0$, $p_0 \neq 0$, $n \geq m$, belong to $\mathrm{DRat}_{SM}(n;m)$. Consider the map, $V:\overline{\mathrm{DRat}}_{SM}(\hat{n};\hat{m}) \to R^1$ defined by

$$\hat{g}(z) \to \frac{1}{2\pi i} \oint_C \frac{g(z)\,g(z^{-1})}{\hat{g}(z)\,\hat{g}(z^{-1})} \frac{dz}{z} = V$$

Then

$$V(\hat{q},\hat{p}) \geq 1 \quad \text{and}$$

$$V(\hat{q},\hat{p}) = 1 \quad \text{iff}$$

$$(*) \qquad q(z)\,\hat{p}(z) = \hat{q}(z)\,p(z).$$

Further,

(a) Equ (*) has a solution iff $\min(\hat{n}-n, \hat{m}-m) \geq 0$.

(b) If $\hat{n}=n$ and $\hat{m} \geq m$ or $\hat{n} \geq n$ and $\hat{m}=m$, then (*) has a unique solution which is the global minimum of V on $\overline{\mathrm{DRat}}_{SM}(\hat{n};\hat{m})$.

(c) If $\min(\hat{n}-n, \hat{m}-m) \geq 0$, then the only critical pts. of V are the solutions of (*).

What we notice here is that in searching for a maximum likelihood estimate in $\overline{\mathrm{DRat}}_{SM}(\hat{n};\hat{m})$ with $\hat{n}>n$ and $\hat{m}>m$ we never find the best approximation within $\mathrm{DRat}_{SM}(\hat{n};\hat{m})$. For this it is necessary to fix $(\hat{n};\hat{m})$ and let $\mathrm{DRat}_{SM}(\hat{n};\hat{m})$ be the manifold on which to search for min(V). This is also required since for $\hat{n}<n$ or $\hat{m}<m$, critical points of V need not be minima. Further, the Fisher-Riemann metric [1] defined by

$$F_{pq} = \left. \frac{\partial^2 V}{\partial \hat{\theta}^p \, \partial \hat{\theta}^q} \right|_{\hat{\theta} = \theta},$$

is degenerate at common factors (this is a consequence of the Morse lemma [10] which requires nondegenerate critical pts. to be isolated) causing unidentifiability. Our solution to the problem is to treat $\mathrm{DRat}_{SM}(n;m)$ or Rat(n) as the parameter space and carry out Bayesian inference as follows:

Maximize (Posterior Support)

Where Posterior support = log (likelihood) + log (p(g)) where p(g) the prior density satisfies the nondegneracy condition that it is nonzero on Rat(n) and vanishes at commonfactors (i.e. $p(g) \to 0$ as resultant $(g) \to 0$). Any algorithm to search for the maximum of the posterior support will remain within connected components because of the nondegeneracy condition. One natural candidate for a prior is the volume element associated with the Fisher-Riemann metric. Variational priors in general [4, 11] belong to this category. Densities that admit invariant Bayesian inference are the subject of the rest of this paper.

3. INVARIANTS OF SCALINGS

Let

$$g(s) = \frac{q(s)}{p(s)} = \frac{q_{n-1} s^{n-1} + q_{n-2} s^{n-2} + \dots + q_1 s + q_0}{s^n + p_{n-1} s^{n-1} + \dots + p_1 s + p_0}$$

Consider the Bezoutian form $B(q,p) = q(x)p(y) - q(y)p(x)$. Since for

$x = y$, $B(q,p) = 0$, $(x-y)$ divides the above form. Therefore, we have

$$\tilde{R}(q,p;\, x,y) = (q(x)\,p(y) - q(y)\,p(x))/(x-y)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} c_{ik}\, x^{i-1} y^{k-1} \qquad c_{ik} = c_{ki}$$

Then it can be shown that the resultant of q and p is given by [12].

$$\underset{\sim}{R} = \mathrm{Res}(q,p) = \det[c_{ik}]$$

Thus $g(s) \in \mathrm{Rat}(n)$ iff the quadratic form $[c_{ik}]$ is nondegenerate. Firstly we view the resultant as a nonvanishing function on Rat(n) (a differential form of degree 0).

Def^{n}3: A differential form ω on a manifold M is said to be a relative invariant under the action of a group G on M if there exists a function $\psi: G \to R^1$ such that $\omega(gm) = \psi(g)\cdot\omega(m)$ $\forall g \in G$ and $m \in M$. Further, it is an absolute invariant if $\psi(g) = 1$ $g \in G$. We have,

Lemma 2: The resultant R is a relative invariant of the scaling group G_A satisfying the following transformations:

(1) $s \to \alpha s$ $\quad (\alpha \in R^+)$; $\underset{\sim}{R} \to \alpha^{-n^2}\underset{\sim}{R}$

(2) $s \to s + \sigma$ $\quad (\sigma \in R)$; $\underset{\sim}{R} \to \underset{\sim}{R}$

(3) $g(s) \to m\,g(s)$ $\quad (m \in R^+)$; $\underset{\sim}{R} \to m^n\underset{\sim}{R}$

(4) $g(s) \to \dfrac{g(s)}{1 + k\,g(s)}$ $\quad (k \in R)$; $\underset{\sim}{R} \to \underset{\sim}{R}$

Remark: The resultant is not even a relative invariant of the shift $w(t) \to w(t + \tau)$. To see this consider the example:

$$g(s) = \frac{\alpha}{s+\lambda}\ ;\ \underset{\sim}{R} = \alpha. \ \text{ Under } t \to t + \tau,\ \underset{\sim}{R} \to e^{-\lambda\tau}\underset{\sim}{R}.$$

To understand what the other invariants are, consider the infinitesimal feedback scaling (generates $g(s) \to \dfrac{g(s)}{1-tg(s)}$).

$$(+) \qquad \frac{dq_i}{dt} = 0;\ \frac{dp_i}{dt} = -q_i \qquad i=0, 1, 2, .., n-1 .$$

This is a Hamiltonian system in R^{2n} with

$$H = H(q,p) = \sum_{i=0}^{n-1} q_i^{\,2}/2$$

To describe this in Rat(p,q), recall that R^{2n} carries a canonical symplectic structure [5],

$$\omega = \Sigma\, dq_i \wedge dp_i$$

and if f denotes the natural imbedding $f : \mathrm{Rat}(n) \to R^{2n}$ then $\omega_F = f^*\omega$ is the natural pullback closed 2-form on Rat(p,q). Denoting by,

$$\lambda_F = \Sigma - q_i \frac{\partial}{\partial p_i}$$

the feedback vectorfield on Rat(p,q) we see that (+) is a statement of the invariance,

$$(\text{Lie derivative wrt}\lambda_F) \quad D_{\lambda_F}\,\omega_F = 0$$

This immediately implies,

$$(++) \qquad D_{\lambda_F}\,\Omega = 0$$

where $\Omega = (\omega_F)^{\wedge n}$ is the canonical 2n form. Thus, we see from (++) that Rat(n) carries a canonical smooth invariant (under feedback) measure with density Ω. This definitely does not satisfy the nondegeneracy condition of Section 3. However the associated 2n form

$$\Omega_1 = \underset{\sim}{R} \cdot \Omega$$

is a λ_F-invariant volume element (from Lemma 2) and also satisfies the nondegeneracy condition.

The symplectic structure ω_F is not the only possible one. Consider, for example,

$$g(s) = \sum_{i=1}^{n} \frac{e^{\alpha_i}}{s+\lambda_i} \ \in \mathrm{Rat}(n,o)$$

$\lambda_i \neq \lambda_j$ for $i \neq j$ and α_i real. Under the shift $t \to t + \tau$

$$g(s) \to \Sigma \frac{e^{\alpha_i - \lambda_i\tau}}{s+\lambda_i} .$$

The corresponding infinitesimal representation

$$(+) \qquad \frac{d\alpha_i}{dt} = \lambda_i\ ;\ \frac{d\lambda_i}{dt} = 0 \qquad t = -\tau$$

This is also a Hamiltonian system with $H = \frac{1}{2}\sum_i \lambda_i^2$.

The associated (real) symplectic structure ω_s on Rat(n,o) is $\omega_s = \Sigma\, d\alpha_i \wedge d\lambda_i$ and the canonical smooth shift-invariant density is given by $\Omega_s = (\omega_s)^{\wedge n}$

Remark: (1) ω_s and ω_F are distinct in that there is no symplectic diffeomorphism carrying one to the other.

(2) The flow (+) and the form ω_s coincide with Moser's representation of the Toda lattice [13] and this suggests natural generalizations of isospectral deformations.

In this section we have seen two densities Ω, Ω_1 on Rat(n) associated with a natural symplectic structure. We also noted that the Fisher-Riemann metric given rise to a smooth density. In the next section these ideas are unified in one common setting.

4. INTEGRAL GEOMETRY ON Rat(n)

The general approach of integral geometry is as follows: Let M be a manifold of dimension n.

(a) Find $\omega_1, \omega_2, \ldots, \omega_n$ a set of independent differential 1-forms on M, which are invariant wrt a Lie group G acting on M, i.e. if X is an element belonging to the Lie algebra $\tilde{G}$ of G mapped into an element $\bar{X} \in U(M)$ the set of C^∞ vectorfields on M then

$$D_{\bar{X}}\,\omega_i = 0 \quad i = 1, 2, \ldots n .$$

(b) Then, $\Omega = \bigwedge_{i=1}^{n} \omega_i$ is an invariant density.

Unless M is compact such densities are not probability densities. However, the set function $P(Q/R) = \int_{Q\cap R} \omega \,/ \int_R \omega$ defines a conditional probability structure, where (Q,α) and (R,β) belong to some G-atlas [5] on M. We then have the following natural generalization of a conditional Poisson process:

(1) $P_n(Q/R) = \exp(-\int_{Q\cap R}\omega) \cdot (\int_{Q\cap R}\omega)^n/n! \quad n=0, 1, 2, \ldots$

Where $P_n(Q/R)$ denotes the probability of n Poisson events occuring in the patch (Q,α) with the conditioning that the probability of an event occuring outside the patch (R,β) is zero. If a Riemannian structure is specified on M, then a 1-parameter family of probability densities on M is defined by the fundamental solution to the diffusion equation:

(2) $$\frac{\partial u}{\partial t} = \Delta u$$

Where the Laplace-Beltrami operator,

(3) $$\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^p} \left\{ g^{pq} \sqrt{g} \frac{\partial}{\partial x^q} \right\}$$

Where $g = |\det(g_{pq})|$; $g^{pq} g_{qr} = \delta^p_r$. In general it is hard to find (g_{pq}) admitting a specified group (such as G_A or G_B) as a group of isometries. We consider several cases below.

(a) Probability Density on Rat(n,o): Recall that any $g(s) \in$ Rat(n,o) has a (symmetric) realization [A,b,b'] where A=A' has distinct (real) eigenualues and [A,b] is a cyclic pair. If [F,g,g'] is another such then there is an unique $M \in O(n)$ such that $F = MAM'$; $g = Mb$. Denoting as $\hat{R}_S$ the set of such realizations we have on $\hat{R}_S$ the metric.

(4) $ds^2 = tr(dA^2) + \langle db, db \rangle$

invariant under $O(n) \times \{\pm 1\} \times R^{n(n+3)/2}$ represented by $A \to MAM'$; $b \to Mb$; $A \to A+R$; $b \to b+\ell$; $R=R'$, $\ell \in R^n$ and $M \in O(n)$. Upto scale factor, the corresponding

(5) $$\Delta = \frac{1}{2} \sum_{i\leq n} \frac{\partial^2}{\partial a_{ii}^2} + \frac{1}{4} \sum_{i<j} \frac{\partial^2}{\partial a_{ij}^2} + \frac{1}{2} \sum_i \frac{\partial^2}{\partial b_i^2}$$

The corresponding density is given by

(6) $$p(A,b) \sim \exp\left(\frac{-trA^2}{2t}\right) \quad \exp\left(-\frac{\langle b,b \rangle}{2t}\right)$$

which induces the following density on poles

(7) $$p(\lambda_1, \lambda_2, \ldots, \lambda_n) \sim \exp\left(-\frac{1}{2} \sum_i \lambda_i^2\right) \cdot \prod_{i<j} |\lambda_i - \lambda_j| \qquad \lambda_i \in R$$

Note that it vanishes at repeated poles as it should since in Rat(n,o), repeated poles are inadmissible. However, the cyclicity of [A,b] is not taken into account. The density (7) first arose in Physics in the study of statistics of energy levels and is discussed by McKean [14]). A more natural metric from a system theoretic viewpoint appears in treating the general case.

(b) Riemannian Structure on Minimal Realizations: Let $\hat{R}$ denote the set of all minimal triples [A,b,c]. Let the transfer function $g(s) = c(sI-A)^{-1}b$. Further, let M and N be respectively the controllability and observability matrices. $M = [b, Ab, \ldots, A^{n-1}b]$; $N = [c; cA; \ldots; cA^{n-1}]$. Under R^n change of basis $x \to Px$, $P \in G\ell(n;R)$, we have, $A \to PAP^{-1}$; $b \to Pb$; $c \to cP^{-1}$; $M \to PM$ and $N \to NP^{-1}$. Consider the quadratic differential form,

(8) $$ds^2 = tr(dAdA') + \langle M'dc, M'dc \rangle + \langle Ndb, Ndb \rangle$$

Equ. (8) defines a $G\ell(n)$ invariant Riemannian metric on $\hat{R}$ and the associated volume element is

(9) $$d\mu = \sqrt{g} \prod_{ij} dA_{ij} \prod_k db_k \prod_\ell dc_\ell$$

where $g = \det(MM')\det(NN') = \det(H)$ where H is the system Hankel matrix. But $\det(H) = |Res(q,p)|$ (see [4]). Hence, the density associated with (8) satisfies the nondegeneracy condition and this is true for the solutions of the diffusion equation also. In [4] the projection of the corresponding Laplace operator onto Rat(n) is considered.

(c) Riemannian Structure on Rat(n): Let N be as defined in part (b) above. Consider the set of 2n differential forms of degree 1, defined by

$$\omega_i = dq_{i-1} \qquad i=1, 2, \ldots, n$$

$\omega_{n+1} \cdots \omega_{2n}$ defined by

$$\begin{bmatrix} \omega_1 \\ \cdot \\ \cdot \\ \cdot \\ \omega_n \end{bmatrix} = N \cdot \begin{bmatrix} dp_0 \\ dp_1 \\ \cdot \\ dp_{n-1} \end{bmatrix}$$

As a consequence of observability the $\{\omega_i\}$ span $\Omega^1(Rat(n))$ and a metric $ds^2 = \omega_i \cdot \omega_i$ is available. The corresponding volume element $\omega = \bigwedge_i^{R} dq_i \bigwedge_i dp_i$. The interesting thing is that on the set of all pole models of the form $g(s) = 1 / (s^n + p_{n-1} s^{n-1} + .. + p_0)$, ds^2 above is flat. Further ω is a relative invariant under G_A.

5. CONCLUSIONS

We have described here some preliminary results on geometric methods in identification.

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