Journal of Dynamics and Differential Equations, Vol. 1, No. 3, 1989

The Dynamics of Coupled Planar Rigid Bodies. II. Bifurcations, Periodic Solutions, and Chaos

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Received July 22, 1988

We give a complete bifurcation and stability analysis for the relative equilibria of the dynamics of three coupled planar rigid bodies. We also use the equivariant Weinstein–Moser theorem to show the existence of two periodic orbits distinguished by symmetry type near the stable equilibrium. Finally we prove that the dynamics is chaotic in the sense of Poincaré–Birkhoff–Smale horseshoes using the version of Melnikov's method suitable for systems with symmetry due to Holmes and Marsden.

KEY WORDS: Geometric mechanics; reduction; stability; chaos; rigid body dynamics; periodic orbits.
AMS SUBJECT CLASSIFICATION: 58F.

1. INTRODUCTION

In Part I of this paper (Sreenath *et al.*, 1988), hereafter denoted [I], we studied the Hamiltonian structure and equilibria for interconnected planar rigid bodies, with the primary focus being on the case of three bodies coupled with hinge joints. The Hamiltonian structure was obtained by the reduction technique, starting with the canonical Hamiltonian structure in material representation and then quotienting by the group of Euclidean motions. For three bodies, this Hamiltonian structure is as follows (see Fig. 1): the phase space is $P = S^1 \times S^1 \times \mathbb{R}^3$, parametrized by the two joint angles $\theta_{21} =: \phi$ and $\theta_{32} =: \psi$ and three momenta $\mu = (\mu_1, \mu_2, \mu_3)$ [conjugate to the three angular variables $(\theta_1, \theta_2, \theta_3)$ for the three bodies] with the Poisson bracket

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1040-7294/89/0700-0269$06.00/0

$$\{f, g\} = \left(\frac{\partial f}{\partial \mu_1} - \frac{\partial f}{\partial \mu_2}\right)\frac{\partial g}{\partial \phi} - \left(\frac{\partial g}{\partial \mu_1} - \frac{\partial g}{\partial \mu_2}\right)\frac{\partial f}{\partial \phi} + \left(\frac{\partial f}{\partial \mu_2} - \frac{\partial f}{\partial \mu_3}\right)\frac{\partial g}{\partial \psi} - \left(\frac{\partial g}{\partial \mu_2} - \frac{\partial g}{\partial \mu_3}\right)\frac{\partial f}{\partial \psi}. \tag{1.1}$$

This phase space is obtained by first reducing to center of mass coordinates and then eliminating rotations via

$$P = \frac{T^*(S^1 \times S^1 \times S^1)}{S^1} \cong S^1 \times S^1 \times \mathbb{R}^3.$$

Given a Hamiltonian $H(\phi, \psi, \mu_1, \mu_2, \mu_3)$, the evolution equations $\dot{f} = \{f, H\}$ are equivalent to

$$\begin{aligned}
\dot{\mu}_1 &= \frac{\partial H}{\partial \phi}, \\
\dot{\mu}_2 &= \frac{\partial H}{\partial \psi} - \frac{\partial H}{\partial \phi}, \\
\dot{\mu}_3 &= -\frac{\partial H}{\partial \psi}, \\
\dot{\phi} &= \frac{\partial H}{\partial \mu_2} - \frac{\partial H}{\partial \mu_1}, \\
\dot{\psi} &= \frac{\partial H}{\partial \mu_3} - \frac{\partial H}{\partial \mu_2}.
\end{aligned} \tag{1.2}$$

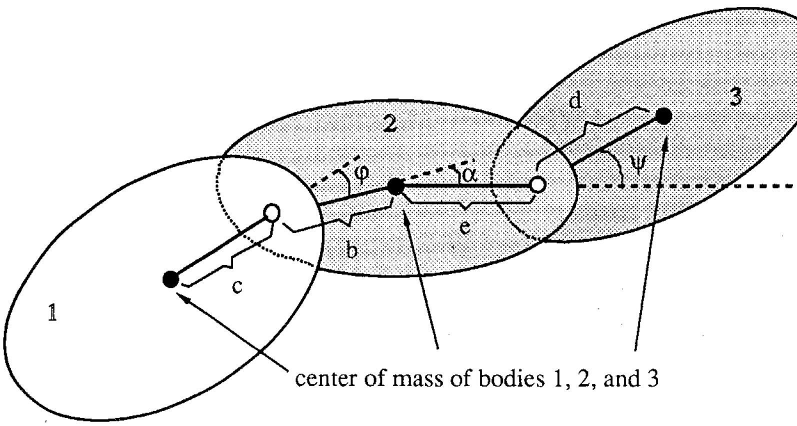

Fig. 1.

For this system, the Hamiltonian is shown in [I] to be

$$H=\tfrac{1}{2}\langle\omega, J\omega\rangle=\tfrac{1}{2}\langle\mu, J^{-1}\mu\rangle \tag{1.3}$$

where

$$(\omega_1, \omega_2, \omega_3)=(\dot\theta_1, \dot\theta_2, \dot\theta_3)=J_\mu$$

is the angular velocity vector and

$$J=\begin{bmatrix} \tilde I_1 & \lambda_{12}(\phi) & \lambda_{31}(\phi+\psi) \\ \lambda_{12}(\phi) & \tilde I_2 & \lambda_{23}(\psi) \\ \lambda_{31}(\phi+\psi) & \lambda_{23}(\psi) & \tilde I_3 \end{bmatrix} \tag{1.4}$$

is the effective moment of inertia matrix; the entries are defined as follows: let c, b, e, d be the positive distances shown in Fig. 1 (i.e., distances between the centers of mass and the hinge points; assume here that the center of mass of the central body is between the hinge points) and let $\varepsilon_{ij}=m_i m_j/(m_1+m_2+m_3)$, I_1, I_2, I_3 be the planar moments of inertia of each body,

$$\tilde I_1=I_1+(\varepsilon_{12}+\varepsilon_{13})\,c^2, \qquad \tilde I_3=(\varepsilon_{23}+\varepsilon_{13})\,d^2,$$
$$\tilde I_2=I_2+(\varepsilon_{12}+\varepsilon_{13})\,b^2+(\varepsilon_{23}+\varepsilon_{13})\,e^2+2\varepsilon_{13}be\cos\alpha$$

be the augmented moments of inertia, and

$$\lambda_{12}(\phi)=(\varepsilon_{12}+\varepsilon_{13})\,bc\cos\phi+\varepsilon_{13}ce\cos(\phi+\alpha),$$
$$\lambda_{31}(\sigma)=\varepsilon_{13}cd\cos\sigma,$$
$$\lambda_{23}(\psi)=(\varepsilon_{23}+\varepsilon_{13})\,de\cos(\psi-\alpha)+\varepsilon_{13}bd\cos\psi.$$

Equilibrium solutions are determined by setting the time derivatives in (1.2) to zero:

$$\begin{aligned} &\frac{\partial H}{\partial\phi}=\frac{\partial H}{\partial\psi}=0, \\ &\frac{\partial H}{\partial\mu_1}=\frac{\partial H}{\partial\mu_2}=\frac{\partial H}{\partial\mu_3}=\omega_0, \qquad \text{a constant.} \end{aligned} \tag{1.5}$$

To further simplify the problem, we will assume that the center of mass of the second body is alligned with the two hinge points; i.e., that $\alpha = 0$ (see Fig. 1). Then (1.5) is equivalent to the system

$$\begin{aligned} \frac{\partial H}{\partial \mu_i} &= \omega_0, \\ \sin(\phi + \psi) &= -\tau \sin \phi, \\ \sin \psi &= \kappa \sin \phi, \end{aligned} \tag{1.6}$$

where

$$\kappa = \frac{\varepsilon_{13}(b+e)c + \varepsilon_{12}bc}{\varepsilon_{13}(b+e)d + \varepsilon_{23}de},$$

and

$$\tau = \frac{\varepsilon_{13}(b+e) + \varepsilon_{12}b}{\varepsilon_{13}d}, \tag{1.7}$$

as was shown in [I].

It was also shown in [I] that there are always four or six equilibria, among which are the four *fundamental equilibria*:

$$(\phi, \psi) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi). \tag{1.8}$$

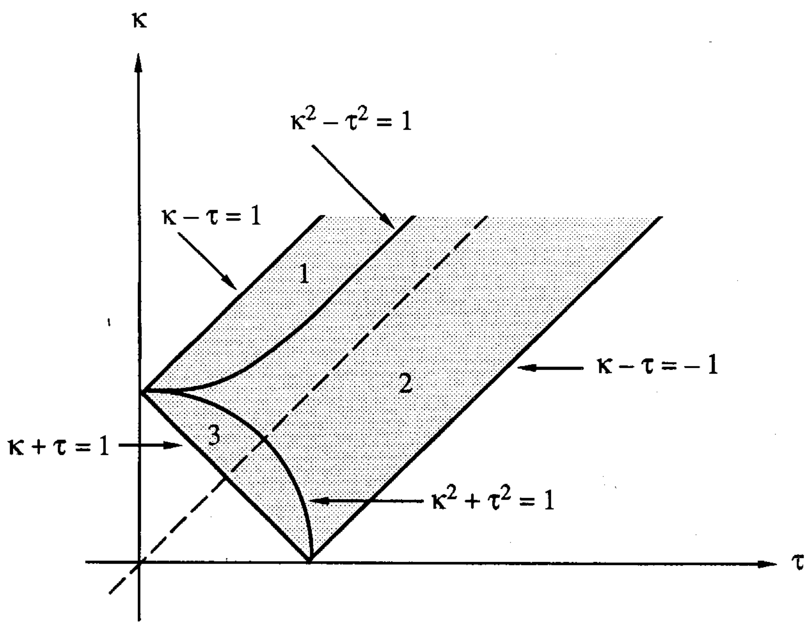

Fig. 2.

When (κ, τ) belongs to the shaded region in Fig. 2, i.e., $|\tau - 1| < \kappa < \tau + 1$, there are two other equilibria determined by

$$\cos\phi = \frac{1 - (\kappa^2 + \tau^2)}{2\kappa\tau} \quad \text{and} \quad \cos\psi = \frac{\kappa^2 - \tau^2 - 1}{2\tau}. \tag{1.9}$$

Correspondingly, the pairs (ϕ, ψ) lie in the shaded region of Fig. 3.

For example, if $m_1 = m_2 = m_3$, $b = e = \mu d$, and $c = \lambda d$, then $\kappa = \lambda$ and $\tau = 3\mu$, so the condition for four equilibria, $|\tau - 1| < \kappa < \tau + 1$, becomes $|3\mu - 1| < \lambda < 3\mu + 1$. For instance, if $\mu = 1$, this condition is $2 < \lambda < 4$. Thus, as λ leaves the range $[2, 4]$, the number of solutions drops from six to four.

In [I], it was shown that the equilibrium $(0, 0)$ representing the straight stretched-out solution is stable for all system parameters. In Section 2, we study bifurcations of these equilibria and we determine the eigenvalue evolution of these bifurcations and thereby determine that the solutions that are not formally stable are not only unstable, but are spectrally and hence exponentially unstable, with nonzero eigenvalues of the linearized equations on the real axis. In Section 3, we use the version of the Weinstein–Moser theorem according to Montaldi *et al.* (1988) to show the existence of two families of periodic orbits (with symmetries) near the stable equilibrium $(\phi, \psi) = (0, 0)$; they are shown to be spectrally stable when they are nonresonant. Finally, in Section 4, we show that the

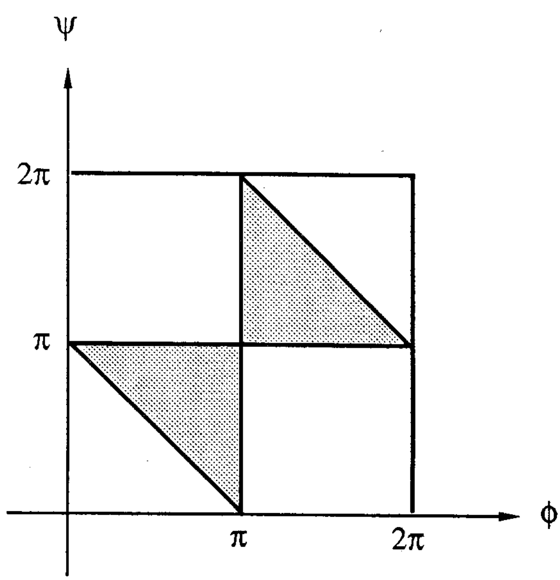

Fig. 3.

problem is, in general, nonintegrable. This is done using the Melnikov method in the version given by Holmes and Marsden (1983) to show that the homoclinic orbit present in the integrable case $d=0$ leads to transverse homoclinic orbits for small $d \neq 0$. General conditions for integrability are not known to us.

We believe that the periodic solutions found in Section 4 are related to traveling waves in a long chain of n-coupled bodies (with torsional springs) or in the corresponding continuum limit $n \to \infty$. This will be the subject of another investigation.

2. BIFURCATION OF EQUILIBRIA

In this section, we relate the bifurcations of equilibria to the degeneracies of the Hessian of the energy function. This is used in the next section where we discuss the stability indices.

First of all, one can see directly from the equilibrium equations, as in [I], that a Hamiltonian pitchfork-type bifurcation occurs at each of the three unstable fundamental equilibrium solutions, as in Fig. 4.

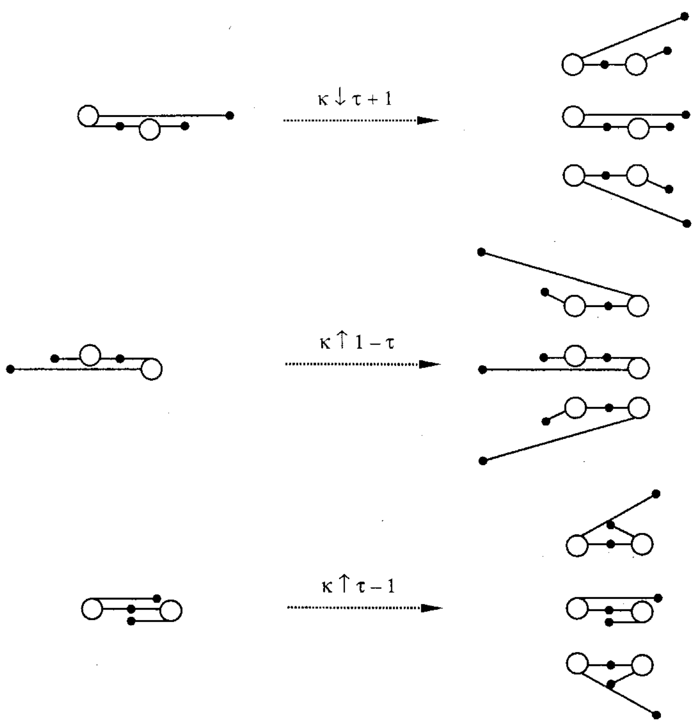

Fig. 4.

For example, if $m_1 = m_2 = m_3$, $c = \lambda d$, and $b = e = d$, the evolution of equilibria as λ ranges from $\lambda < 2$ to $\lambda > 4$ is shown in Fig. 5.

These bifurcations, which can all be seen by direct calculation, will now be related to the second variation, or Hessian, of the Hamiltonian. The symplectic leaves P_μ of the phase space P are defined by setting $\mu_1 + \mu_2 + \mu_3 = \mu$, a constant. Equilibria for the system are exactly critical points of H_μ, the restriction of H to P_μ. At these points, the Hessian of H_μ is simply the restriction of the second variation of H to tangent vectors of P_μ at the equilibrium point in question. Since the Hamiltonian vector field restricted to the leaf has a zero eigenvalue iff the Hessian does, it is a priori clear that a bifurcation of equilibria occurs only if the Hessian along the leaf has a zero eigenvalue.

The Hessian is computed at one of the fundamental equilibria to have the form

$$\delta^2 H = \begin{bmatrix} J^{-1} & 0 \\ 0 & B \end{bmatrix} \tag{2.1}$$

as a 5×5 matrix with the variables in the order $(\mu_1, \mu_2, \mu_3, \phi, \psi)$ restricted to the subspace defined by $\delta\mu_1 + \delta\mu_2 + \delta\mu_3 = 0$, where J is given by (1.4) and where

$$B = (\text{positive constant}) \begin{bmatrix} \cos\phi + \frac{1}{\tau}\cos(\phi+\psi) & \frac{1}{\tau}\cos(\phi+\psi) \\ \frac{1}{\tau}\cos(\phi+\psi) & \frac{1}{\kappa}\cos\psi + \frac{1}{\tau}\cos(\phi+\psi) \end{bmatrix}. \tag{2.2}$$

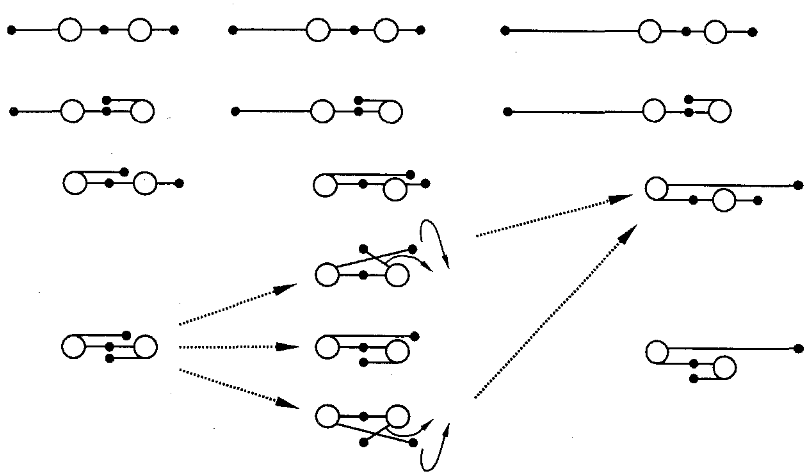

Fig. 5.

Ignoring the positive constant, we note that at the fundamental equilibria,

$$B(0,0)=\begin{bmatrix}\frac{1}{\tau}+1 & \frac{1}{\tau}\\ \frac{1}{\tau} & \frac{1}{\kappa}+\frac{1}{\tau}\end{bmatrix},\qquad B(\pi,0)=\begin{bmatrix}-\frac{1}{\tau}-1 & -\frac{1}{\tau}\\ -\frac{1}{\tau} & \frac{1}{\kappa}-\frac{1}{\tau}\end{bmatrix},\tag{2.3}$$

$$B(0,\pi)=\begin{bmatrix}1-\frac{1}{\tau} & -\frac{1}{\tau}\\ -\frac{1}{\tau} & -\frac{1}{\kappa}-\frac{1}{\tau}\end{bmatrix},\qquad B(\pi,\pi)=\begin{bmatrix}\frac{1}{\kappa}-1 & \frac{1}{\tau}\\ \frac{1}{\kappa} & -\frac{1}{\kappa}-\frac{1}{\tau}\end{bmatrix}.$$

Since J is positive definite, bifurcations of equilibria are determined by zero eigenvalues of B. Since

$$\det B(0,0)=\frac{1}{\kappa}+\frac{1}{\tau}+\frac{1}{\tau\kappa}>0,$$

$(0, 0)$ never bifurcates. Since

$$\det B(\pi,0)=\frac{\kappa-\tau-1}{\kappa\tau},$$

$$\det B(0,\pi)=\frac{1-\kappa-\tau}{\kappa\tau},$$

and

$$\det B(\pi,\pi)=\frac{\tau-\kappa-1}{\kappa\tau},$$

we can expect these equilibria to bifurcate at $\kappa=\tau+1$, $\kappa=1-\tau$, and $\kappa=\tau-1$, respectively. As we saw above, this is confirmed by a direct analysis of the equilibria.

To analyze the stability of these equilibria, notice first that the stretched-out state $(0, 0)$ is always stable, as we already know from [I]. For the state (π, π), note that

$$\det B(\pi,\pi)=\frac{\tau-\kappa-1}{\kappa\tau},\qquad \operatorname{trace} B(\pi,\pi)=\frac{2\kappa-\tau\kappa-\tau}{\tau\kappa},$$

so $B(\pi, \pi)$ has

$\kappa>\tau-1$: one negative and one positive eigenvalue,

$\kappa=\tau-1$: one negative and one zero eigenvalue,

$\kappa<\tau-1$: two negative eigenvalues.

There are similar statements for the equilibria $(\pi, 0)$ and $(0, \pi)$ where $\tau - 1$ is replaced by $\tau + 1$ and $1 - \tau$, respectively.

Theorem 2.1. *The equilibrium $(0, 0)$ is always stable and the other fundamental equilibria are unstable, and in fact spectrally unstable.*

The proof relies on

Lemma 2.1. *Let A and B be two real $n \times n$ symmetric invertible matrices with different numbers of negative eigenvalues. Then the infinitesimally symplectic matrix*

$$\begin{bmatrix} 0 & B \\ -A & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

has at least one positive (and so one negative) real eigenvalue.

Proof of Lemma 2.1

$$\det\left(\lambda I_{2n} - \begin{pmatrix} 0 & B \\ -A & 0 \end{pmatrix}\right) = \det\begin{pmatrix} \lambda I & -B \\ A & \lambda I \end{pmatrix}$$

$$= \lambda^{2n} \det\begin{pmatrix} I & -\frac{1}{\lambda} B \\ \frac{1}{\lambda} A & I \end{pmatrix}$$

(notice that $\lambda \neq 0$ since we assume that A and B are invertible)

$$= \lambda^{2n} \det\begin{pmatrix} I & -\frac{1}{\lambda} B \\ 0 & I + \frac{1}{\lambda^2} AB \end{pmatrix}$$

$$= \lambda^{2n} \det\left(I + \frac{1}{\lambda^2} AB\right) = \det(\lambda^2 + AB).$$

The lemma follows from this sublemma.

Sublemma 2.1. *Under the same hypothesis as Lemma 2.1, the matrix AB has at least one negative eigenvalue.*

Proof of Sublemma 2.1. Since we assume that A is invertible, A^{-1} has the same inertia index (number of negative eigenvalues) as A. Now, consider the 1-parameter family of symmetric matrices $M(t) = tA^{-1} + (1-t)B$, $0 \leqslant t \leqslant 1$. We know that the set of invertible symmetric matrices has $n+1$ components that are characterized by inertia indices. Since $M(0)$ and $M(1)$ have different indices and so are contained in different components and $\{M(t)\}$ is connected, there must be some $0 < t_0 < 1$ for which $M(t_0)$ is not invertible, i.e., there exists a nonzero vector $\mathbf{v}$ such that $M(t_0)\mathbf{v} = 0$, i.e.,

$$(t_0 A^{-1} + (1-t_0)B)\mathbf{v} = 0.$$

Multiplying by A, we get

$$AB\mathbf{v} = -\frac{t_0}{1-t_0}\mathbf{v}.$$

Here, $-t_0/(1-t_0)$ is negative since $0 < t_0 < 1$. Therefore, AB has a negative eigenvalue. ▮

Remarks 2.1. We can refine this lemma to allow one of the matrices not to be invertible. More specifically, let us assume that A is invertible and is of type (p, q) i.e., A has (p positive and q negative eigenvalues) and B is of type (p', q', r) where r is the number of zero eigenvalues. Then AB must have at least one negative eigenvalue if $p > p' + r$ or $q > q' + r$. This then yields Lemma 2.1 as before.

Remark 2.2. Similar criteria for Krein collisions (Hamiltonian Hopf bifurcations) would be of use in the case of *three*-dimensional coupled rigid bodies (cf. Grossman *et al.*, 1988; Cartan, 1928).

Remark 2.3. Some related results and applications are given in Oh (1987).

Lemma 2.2. *In the reduced symplectic space P_μ, the Hessian of the reduced Hamiltonian has the form*

$$\begin{pmatrix} J' & 0 \\ 0 & B \end{pmatrix},$$

where J' is a positive definite 2×2 matrix and B is the matrix (2.2) in the canonical coordinate near the equilibria given by $(\phi, \psi, \nu_1, \nu_2)$, where

$$\nu_1 = \frac{\mu_2 - \mu_1}{2}, \qquad \nu_2 = \frac{\mu_3 - \mu_2}{2}.$$

Proof of Lemma 2.3. This follows from the fact that, at the equilibria, d^2H are given by (2.1) in $T^*(S^1\times S^1\times S^1)/S^1$ which is parametrized by $(\mu_1, \mu_2, \mu_3, \phi, \psi)$ and the fact that

$$\left(\frac{\mu_2-\mu_1}{2}, \frac{\mu_3-\mu_2}{2}, \phi, \psi\right)$$

are canonical coordinates on P_μ near the equilibria; the latter is checked directly from the bracket (1.1) ▮

Proof of Theorem 2.1. We know that, in canonical coordinates, the symplectic structure is given by

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and, by Lemma 2.3, the Hessian has the form

$$\begin{pmatrix} J' & 0 \\ 0 & B \end{pmatrix}$$

where the number of negative eigenvalues of J' is zero and, as we illustrated at $B(\pi, \pi)$, B has at least one negative eigenvalue. Therefore, by Lemma 2.1, the linearization of the Hamiltonian vector has least one negative eigenvalue. Therefore, by Lemma 2.1, the linearization of the Hamiltonian vector field at the equilibria, namely,

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} J' & 0 \\ 0 & B \end{pmatrix},$$

has at least one real eigenvalue and so is spectrally unstable and so is nonlinearly unstable. ▮

Finally, we study the stability of the bifurcation branches at the equilibria. By Theorem 2.1, we know that the linearization of the Hamiltonian vector field at (π, π) [similarly for $(0, \pi)$, $(\pi, 0)$] has at least one and so two real eigenvalues. This came from observing that

$$B(\pi, \pi) = \begin{pmatrix} \frac{1}{\tau} - 1, & \frac{1}{\tau} \\ \frac{1}{\tau}, & \frac{1}{\tau} - \frac{1}{\kappa} \end{pmatrix}$$

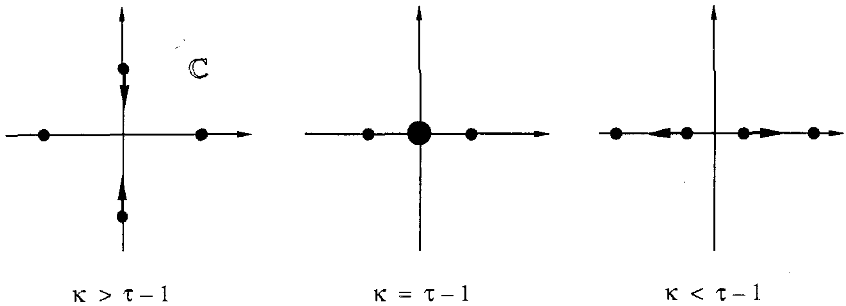

Fig. 6.

has

$$\det B(\pi, \pi) > 0 \quad \text{if} \quad \kappa < \tau - 1,$$
$$\det B(\pi, \pi) < 0 \quad \text{if} \quad \kappa > \tau - 1.$$

Using Lemma 2.1, we get Fig. 6 for the positions of eigenvalues of $DX_H(\pi, \pi)$ with respect to the parameters (τ, κ).

By the spectral property for the eigenvalues of an infinitesimally symplectic matrix, any small perturbation of the $DX_H(\pi, \pi)$ at $\kappa = \tau - 1$ must have real eigenvalues. Hence the DX_H at the bifurcated equilibria must have real eigenvalues at least near the bifurcation parameter. Thus we have the following theorem.

Theorem 2.2. *All of the bifurcation branches from unstable equilibria are themselves unstable.*

3. PERIODIC SOLUTIONS

We have seen that the straight stretched-out state $(\phi, \psi) = (0, 0)$ is stable for all system parameters. Correspondingly, the second variation of the Hamiltonian on the symplectic leaf is positive definite at this point. If ω_0 denotes the angular velocity of this solution, its angular momentum is

$$\mu = \mu_1 + \mu_2 + \mu_3 = \omega_0 \sum_{i,j=1}^{3} J_{ij}(0, 0)$$

and its energy is

$$E_0 = \tfrac{1}{2}\langle \omega_0(1, 1, 1), \omega_0 J(1, 1, 1)\rangle = \tfrac{1}{2}\omega_0^2 \sum_{i,j=1}^{3} J_{ij}(0, 0) \qquad (3.1)$$

where J is the matrix (1.4). The theorem of Weinstein (1973, 1978) and Moser (1976) gives the following:

Theorem 3.1. *For any small $\varepsilon > 0$, there are at least two distinct periodic orbits near this equilibrium on the energy surface $H^{-1}(E_0+\varepsilon)$ in the leaf P_μ.*

These periodic motions in the reduced symplectic manifold produce quasi-periodic motion on a torus in the original phase space $T^*(S^1 \times S^1 \times S^1)$ by S^1 symmetry.

Theorem 3.1 does not directly tell us properties of these periodic orbits, such as their spatial structure. We will obtain such information in the case of a symmetric system by applying an equivariant version of the Weinstein–Moser theorem according to Montaldi *et al.* (1988).

We assume that the 3-body system is symmetric under the transformation of configuration space given by

$$(\theta_1, \theta_2, \theta_3) \mapsto (\theta_3, \theta_2, \theta_1) \qquad \text{in} \quad S^1 \times S^1 \times S^1. \tag{3.2}$$

This means, in effect, that bodies 1 and 3 are mechanically identical. This assumption gives the symmetries $J_{11} = J_{33}$ and $J_{12} = J_{23}$ of the metric J besides J being symmetric. The transformation (3.2) on $S^1 \times S^1 \times S^1$ induces a $\mathbb{Z}_2$-action on the phase space $T^*(S^1 \times S^1 \times S^1)$ and the Hamiltonian (= the kinetic energy) is invariant under this action; this is our symmetry assumption. Obviously, this $\mathbb{Z}_2$-action commutes with the diagonal S^1-action and so it induces a symplectic $\mathbb{Z}_2$-action on the reduced space P_μ. In canonical coordinates (ϕ, ψ, v_1, v_2) on P_μ where $\phi = \theta_2 - \theta_1$, $\psi = \theta_3 - \theta_2$, $v_1 = (\mu_2 - \mu_1)/2$, and $v_3 = (\mu_3 - \mu_2)/2$, this $\mathbb{Z}_2$-action can be written as

$$(\phi, \psi, v_1, v_2) \to (-\psi, -\phi, -v_2, -v_1). \tag{3.3}$$

Its *fixed manifold* is given by

$$F = \{(\theta_{21}, \theta_{32}, v_1, v_2) \mid \phi + \psi = 0 \ (\mathrm{mod}\ 2\pi)\ v_1 + v_2 = 0\}. \tag{3.4}$$

We have the following general facts about fixed manifold under symplectic actions:

Proposition 3.1. *Let a compact Lie group G act symplectically on a symplectic manifold P. Then each component of the fixed point set* Fix(G) *is a symplectic submanifold of P.*

Proof. See, for instance, Guillemin and Sternberg (1984). ∎

Proposition 3.2. *Let $H: P \to \mathbb{R}$ be a G-invariant Hamiltonian and let X_H be the associated Hamiltonian vector field. Then X_H is tangent to each component of* $\mathrm{Fix}(G)$ *and* $X_H \mid \mathrm{Fix}(G)$ *has the Hamiltonian* $H \mid \mathrm{Fix}(G)$.

Proof. See Golubitsky and Stewart (1987). ▌

From these propositions, we see that F is a symplectic submanifold (in fact, one can easily check this without referring to Proposition 3.1) and the restriction of the Hamiltonian H will give a Hamiltonian on F. Notice that F is actually diffeomorphic to T^*S^1 which is two-dimensional, and so the dynamics on it is completely integrable. From the general fact that zeros of X_H in F are zeroes of $X_{H \mid F}$, we can see that $X_{H \mid F}$ has *two* zeroes and corresponding to $(\phi, \psi) = (0, 0)$ and (π, π) among the four fundamental equilibria.

Focusing on the induced dynamics on $F \cong T^*S^1$, notice that any level surfaces of the reduced Hamiltonian in F are compact since those of the original Hamiltonian H are. Since we have proved that the induced Hamiltonian vector field $X_{H \mid F}$ has one stable equilibrium and one unstable one, the dynamics is qualitatively similar to the reduced dynamics of the coupled 2-body case. In particular, we conclude that the original Hamiltonian system in P_μ has infinitely many periodic orbits and at least two homolinic orbits.

We summarize the above discussions:

Theorem 3.2. *For the symmetric coupled planar 3-body system, every symmetric initial condition gives rise to a symmetric periodic motion (up to diagonal action by S^1) unless the initial energy is the same as the energy of the two equilibria* $(0, 0)$ *and* (π, π). *Moreover, the energy surface of the unstable equilibrium* (π, π) *contains two homoclinic orbits issuing from it.*

From this theorem, we may expect that the slightly unsymmetric system will have chaotic phenomena. This may be proved by an adaptation of the Melnikov method (see Section 4).

We consider the dynamics on the energy surface $H^{-1}(E_0 + \varepsilon)$ for small $\varepsilon > 0$. We already know from Theorem 3.1 that this level surface contains at least two distinct periodic orbits. To get information on their symmetry, we use the following:

Theorem 3.3 *[Equivariant Weinstein–Moser Theorem (Montaldi et al., 1988)]. Let G be a group acting symplectically on a symplectic manifold (P, Ω) and H be an invariant Hamiltonian. Let $z \in P$ be a fixed point for the corresponding Hamiltonian vector field and assume:*

H1 *the Hesian $d^2H(z)$ is nondegenerate; and*

H2 *$d^2H(z_0)$ restricted to a resonance subspace V_λ is positive definite. [V_λ is the subspace of T_zP that is the real part of the direct sum of all the generalized eigenspaces of eigenvalues of $L = \mathbf{D}X_H(z)$ that are multiples of the purely imaginary eigenvalue λ.]*

Then for every isotropy subgroup Σ of the $G \times S^1$ action on V_λ, and for $\varepsilon > 0$ sufficiently small, there are at least

$$\tfrac{1}{2} \dim \operatorname{Fix}(\Sigma, V_\lambda) \tag{3.5}$$

periodic trajectories of X_H with periods near $2\pi/|\lambda|$ and symmetry group containing Σ, on the energy surface $H = E_0 + \varepsilon$.

To apply this theorem, we need information about the eigenvalues and generalized eigenspace of the linearization $\mathbf{D}X_H(z)$ of the Hamiltonian vector field at the stable equilibrium. The Hamiltonian is quite complicated and so it is tedious to find the eigenvalues and eigenspace directly. Fortunately, we do not have to do this. Instead, we will fully exploit the $\mathbb{Z}_2$-symmetry and solve using general facts about symplectic representations (see Guillemin and Sternberg, 1984). We identify the tangent space to P_μ at the stable equilibrium with $\mathbb{C}^2$ by setting

$$z_1 = \dot{\phi} + iv_1 \qquad \text{and} \qquad z_2 = \dot{\psi} + iv_2. \tag{3.6}$$

Then the induced $\mathbb{Z}_2$-representation on this tangent space is decomposed into irreducible representations; $\mathbb{C}^2 = \mathbb{C}_0 \oplus \mathbb{C}_1$ where $\mathbb{C}_0$ is the trivial piece and $\mathbb{C}_1$ is the nontrivial piece. In fact, $\mathbb{C}_0 = \{(z_1, z_2) \mid z_1 + z_2 = 0\}$, which is the tangent space to F at the equilibrium and $\mathbb{C}_1 = \{(z_1, z_2) \mid z_1 - z_2 = 0\}$.

The group $\mathbb{Z}_2$ acts on $\mathbb{C}_0$ trivially and on $\mathbb{C}_1$ by $(z, z) \mapsto (-z, -z)$. Since Hessian of H at this equilibrium is positive definite, all eigenvalues of the linearization of X_H are imaginary and come in pairs $\pm i\lambda_1$, $\pm i\lambda_2$ where λ_1 and λ_2 may be the same. Since $\mathbb{C}_0$ is the tangent space to F, the linearization will have $\mathbb{C}_0$ as a generalized eigenspace corresponding to, say, $\pm i\lambda_1$. Since we know that each generalized eigenspace is symplectic and pairwise orthogonal, $\mathbb{C}_1$ will be the generalized eigenspace of $\pm i\lambda_2$.

We summarize the above discussions:

Proposition 3.3. *The tangent spaces at the stable equilibrium identified with $\mathbb{C}^2$ by (3.6) are decomposed into irreducible pieces of the induced representation of $\mathbb{Z}_2$;*

$$\mathbb{C}^2 = \mathbb{C}_0 \oplus \mathbb{C}_1$$

where

$$\mathbb{C}_0 = \{(z_1, z_2) \mid z_1 + z_2 = 0\} \quad \textit{and} \quad \mathbb{C}_1 = \{(z_1, z_2) \mid z_1 - z_2 = 0\}. \quad (3.7)$$

Moreover, these irreducible components correspond to generalized eigenspaces of the linearization of X_H with the eigenvalues $\pm i\lambda_1$, $\pm i\lambda_2$, respectively, $\lambda_1, \lambda_2 > 0$.

From this proposition, we conclude that the flows of the linearization L on $\mathbb{C}_0$ and $\mathbb{C}_1$ are equivalent to

multiplication by $e^{2\pi i s\lambda_1}$ on $\mathbb{C}_0$ and multiplication by $e^{2\pi i s\lambda_2}$ on $\mathbb{C}_1$. (3.8)

Thus, we have $\mathbb{Z}_2 \times S^1$ actions on $\mathbb{C}_0$ and $\mathbb{C}_1$, respectively.

Next, we find the isotropy groups of these actions on each of $\mathbb{C}_0$ and $\mathbb{C}_1$:

On $\mathbb{C}_0$, the isotropy group is $\mathbb{Z}_2 \times \{1\}$ and whole space $\mathbb{C}_0$ is the fixed point space of real dimension 2.

On $\mathbb{C}_1$, the isotropy group is $\{-1\} \times \{-1\}$ and again the whole space $\mathbb{C}_1$ is the fixed point space of real dimension 2.

Therefore, we have the following refinement of Theorem 3.1:

Theorem 3.4. *For any small $\varepsilon > 0$, we have at least one periodic orbit with $\mathbb{Z}_2$-symmetry and at least one periodic orbit with $\{-1\} \times \{-1\}$ symmetry on the energy surface $H^{-1}(E_0 + \varepsilon)$ in P_μ.*

Remarks 3.1. As we mentioned before, these periodic orbits give quasi-periodic orbits in the original phase space $T^*(S^1 \times S^1 \times S^1)$. They have the pictures in Fig. 7 in the stick representation viewed from a rotating frame, i.e., up to the diagonal S^1-action.

Remark 3.2. When $\lambda_1 \neq \lambda_2$, one can apply an equivariant version of the Liapunov Center Theorem to produce smooth families of periodic orbits with corresponding symmetries bifurcating from the stable equilibrium.

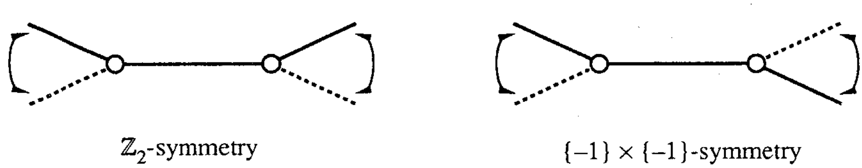

Fig. 7.

Remark 3.3. We conjecture that these periodic solutions are related to traveling waves for many bodies and in the continuum limit.

Finally, in this section, we examine some aspects of stability. If we let ϕ_t be the Hamiltonian flow, then the *Floquet operator* $M(u)$ of a periodic orbit $u(t)$ with period T is defined by

$$M(u) = D\phi_T(u(0)) \colon T_{u(0)}P \to T_{u(0)}P. \tag{3.9}$$

If $M(u)$ has all eigenvalues on the unit circle, then u is called *spectrally stable.* Note that $M(u)$ always has a generalized eigenspace of dimension at least 2 with eigenvalue 1 because

$$u'(T) = D\phi_T(u(0)) \cdot u'(0) = u'(0). \tag{3.10}$$

Now, let u_1 and u_2 be the periodic solutions in Theorem 3.4, whose periods T_1 and T_2 are near $2\pi/|\lambda_1|$ and $2\pi/|\lambda_2|$, respectively. Then we have the following result about the spectral stability.

Theorem 3.5. *If λ_1 and λ_2 are nonresonant, then the two periodic orbits that were found in Theorem 3.4 are spectrally stable if $\varepsilon > 0$ is small.*

Proof. Note that the Floquet operator $M(u_i)$ is close to exp $(-T_i \mathbf{D}X_H(z))$ as $\varepsilon \to 0$. Note that $T_i \approx 2\pi/|\lambda_i|$ and $\exp((-2\pi/|\lambda_i|)\,\mathbf{D}X_H(z))$ has eigenvalue whose corresponding generalized eigenspace is of dimension 2 and so has one simple eigenvalue pair that lies on the unit circle. By the general rigidity of the behavior of the eigenvalues of perturbations of a symplectic matrix, we conclude that the eigenvalues of $M(u_i)$ stay on the unit circle if $\varepsilon > 0$ is small. ▌

4. CHAOTIC SOLUTIONS

In this section, we show that the dynamics of the three coupled rigid body system is not integrable, having chaotic solutions of horseshoe type. This is done using the Holmes and Marsden (1983) version of Melnikov's method to perturb a homoclinic orbit in a problem with S^1 symmetry. There are several homoclinic orbits that one can use to perturb; a pair was described in Theorem 3.2. Here we perturb the two body problem by adding a third body near the center of mass of the second.

We first need to derive an expression for the Hamiltonian that is written so the perturbing terms are isolated. Refer to Fig. 8.

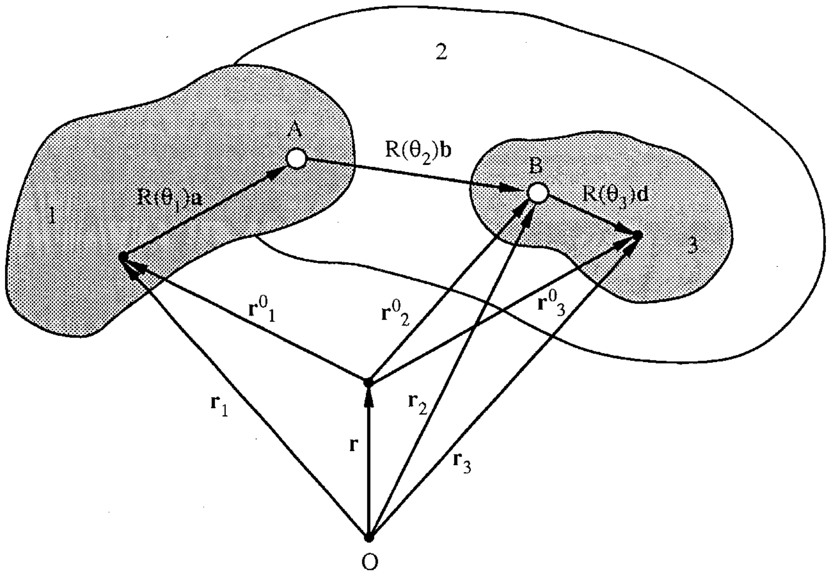

Fig. 8.

Let

O be the origin of the inertial reference frame;

A be the hinge point of bodies 1 and 2;

B be the hinge point of bodies 2 and 3, and also the center of mass of body 2;

$\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$ be the vectors between the centers of mass and hinge points of bodies 1, 2, 3 in the reference configuration;

$R(\theta)$ be the rotation through angle θ;

$\mathbf{r}$ be the vector from O to the system center of mass;

$\mathbf{r}_1^0$, $\mathbf{r}_2^0$, $\mathbf{r}_3^0$ be vectors from the system center of mass to the body centers of mass;

θ_1, θ_2, θ_3 be rotation angles from a reference configuration to the current configuration;

$\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ be position vectors for points in bodies 1, 2, 3 in the reference configuration; and

$\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ be position vectors for points in bodies 1, 2, 3 in the current configuration.

As in [I], we have

$$\mathbf{x}_i = R(\theta_i)\mathbf{X}_i + \mathbf{r}_i, \tag{4.1a}$$

$$\mathbf{r}_2 = \mathbf{r}_1 + R(\theta_1)\mathbf{a} + R(\theta_2)\mathbf{b}, \tag{4.1b}$$

$$\mathbf{r}_3 = \mathbf{r}_2 + R(\theta_3)\mathbf{d}, \tag{4.1c}$$

$$m\mathbf{r} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3. \tag{4.1d}$$

We compute the total kinetic energy as in [I] as

$$\begin{aligned} H = {} & \sum_{i=1}^{3} \frac{1}{2}\operatorname{trace}(\omega_i I^i \omega_i^T) + \frac{p^2}{2m} \\ & + \frac{1}{2} m_1 \left\| -\frac{m_2+m_3}{m}(\dot R_1\mathbf{a} + \dot R_2\mathbf{b}) - \frac{m_3}{m}\dot R_3\mathbf{d} \right\|^2 \\ & + \frac{1}{2} m_2 \left\| \frac{m_1}{m}(\dot R_1\mathbf{a} + \dot R_2\mathbf{b}) - \frac{m_3}{m}\dot R_3\mathbf{d} \right\|^2 \\ & + \frac{1}{2} m_3 \left\| \frac{m_1}{m}(\dot R_1\mathbf{a} + \dot R_2\mathbf{b}) - \frac{m_1+m_3}{m}\dot R_3\mathbf{d} \right\|^2. \end{aligned} \tag{4.2}$$

Assume $p = 0$, without loss of generality, and that the reference configuration is chosen so $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{d}$ are parallel. Then we can write

$$H = H_d = H_0 + dH_1 + O(d^2) \tag{4.3}$$

where $d = \|\mathbf{d}\|$ is our small parameter,

$$H_0 = \tfrac{1}{2}(I_1 + \varepsilon a^2)\,\omega_1^2 + \tfrac{1}{2}(I_2 + \varepsilon b^2)\,\omega_2^2 + \varepsilon ab\cos\phi\,\omega_1\omega_2 + \tfrac{1}{2}I_3\omega_3^2, \tag{4.4a}$$

and

$$H_1 = \gamma(a\cos(\phi+\psi)\,\omega_1\omega_3 + b\cos\psi\,\omega_2\omega_3) \tag{4.4b}$$

where

$$\varepsilon = \frac{m_3(m_1+m_2)}{m}, \qquad \gamma = \frac{m_1 m_3}{m}, \qquad \text{and} \qquad O(d^2) = \frac{m_3(m_1+m_2)}{m} d^2\omega_3^2.$$

We can rewrite H as $H = \frac{1}{2}\langle \omega, J_d\omega\rangle$ where $\omega = (\omega_1, \omega_2, \omega_3)^T$,

$$J_d = \begin{pmatrix} \tilde I_1 & \varepsilon ab\cos\phi & \gamma ad\cos(\phi+\psi) \\ \varepsilon ab\cos\phi & \tilde I_2 & \gamma bd\cos\psi \\ \gamma ad\cos(\phi+\psi) & \gamma bd\cos\psi & \tilde I_3 \end{pmatrix} \tag{4.5}$$

and

$$\tilde I_1 = I_1 + \varepsilon a^2, \qquad \tilde I_2 = I_2 + \varepsilon b^2, \qquad \tilde I_3 = I_3 + \frac{m_3(m_1+m_2)}{m} d^2.$$

Write $J_d = J_0 + dJ_1 + O(d^2)$ where

$$J_0 = \begin{pmatrix} \tilde{I}_1 & \varepsilon ab \cos\phi & 0 \\ \varepsilon ab \cos\phi & \tilde{I}_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

and

$$J_1 = \begin{pmatrix} 0 & 0 & \gamma a \cos(\phi+\psi) \\ 0 & 0 & \gamma b \cos\psi \\ \gamma a \cos(\phi+\psi) & \gamma b \cos\psi & 0 \end{pmatrix}.$$

We need to write the kinetic energy with respect to the momentum μ rather than the angular velocity. This is done using $\mu = J_d \omega$:

$$H_d(\mu) = \tfrac{1}{2}\langle \mu, J_d^{-1}\mu\rangle = H_0(\mu) + dH_1(\mu) + O(d^2)$$

where

$$H_1(\mu) = \frac{\partial H_d}{\partial d}\bigg|_{d=0} = -\left\langle \mu, J_0^{-1}\frac{\partial J}{\partial d}\bigg|_{d=0} J_0^{-1}\mu\right\rangle = -\langle \mu, J_0^{-1}J_1J_0^{-1}\mu\rangle. \quad (4.6)$$

Here,

$$J_0^{-1} = \begin{pmatrix} \frac{1}{\Delta}\begin{pmatrix} \tilde{I}_2 & -\varepsilon ab\cos\phi \\ -\varepsilon ab \cos\phi & \tilde{I}_1 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & I_3^{-1} \end{pmatrix}$$

(where $\Delta = \tilde{I}_1\tilde{I}_2 - \varepsilon^2 a^2 b^2 \cos^2\theta$), and so $J_0^{-1}J_1J_0^{-1}$ becomes

$$\begin{bmatrix} 0 & 0 & \frac{\gamma I_3^{-1}}{\Delta}(\tilde{I}_2 a\cos(\phi+\psi) - \varepsilon b^2 a \cos\phi\cos\psi) \\ 0 & 0 & \frac{\gamma I_3^{-1}}{\Delta}[(-\varepsilon a^2 b\cos(\phi+\psi)\cdot\cos\phi) + \tilde{I}_1 b\cos\psi)] \\ \frac{\gamma I_3^{-1}}{\Delta}(\tilde{I}_2 a\cos(\phi+\psi) - \varepsilon b^2 a\cos\phi\cos\psi) & \frac{\gamma I_3^{-1}}{\Delta}[(-\varepsilon a^2 b\cos(\phi+\psi)\cdot\cos\phi) + \tilde{I}_1 b\cos\psi)] & 0 \end{bmatrix}.$$

Therefore,

$$H_0 = \frac{1}{2\Delta}(\tilde{I}_2\mu_1^2 + \tilde{I}_1\mu_2^2 - 2\varepsilon ab\cos\phi\mu_1\mu_2) + \frac{1}{2}I_3^{-1}\mu_3^2 \quad (4.7a)$$

and

$$H_1 = -\frac{\gamma I_3^{-1} a}{\Delta} \mu_1 \mu_3 (\tilde{I}_2 \cos(\phi + \psi) - \varepsilon b^2 \cos\phi \cos\psi)$$

$$- \frac{\gamma I_3^{-1} b}{\Delta} \mu_2 \mu_3 (-\varepsilon a^2 \cos(\phi + \psi) \cos\phi + \tilde{I}_1 \cos\psi). \qquad (4.7b)$$

Now, notice that, when $d = 0$, i.e., B coincides with the center of mass of body 3, the system is completely integrable and we know that the reduced system has two homoclinic orbits, given in Fig. 1 of [I]. This system is in the framework of the Melnikov method with an S^1 symmetry as generalized by Holmes and Marsden (1983).

From (4.7a), we see that the unperturbed Hamiltonian H_0 has an additional S^1 symmetry given by ψ-rotations; or, equivalently, by θ_3-rotations in the original system, which induces the obvious Poisson action whose moment mapping is exactly μ_3: $T^*(S^1 \times S^1 \times S^1)/S^1 \to \mathbb{R}$. Note that $\mu_1 + \mu_2 + \mu_3$ is the moment mapping corresponding to the simultaneous uniform rotation, i.e., the diagonal action of S^1. The H_d and H_0-flows restrict to the symplectic leaf $P_M = \{\mu_1, \mu_2, \mu_3, \phi, \psi \mid \mu_1 + \mu_2 + \mu_3 = M\}$, and (ψ, μ_3) are conjugate variables in this symplectic leaf. Note that the equations of motion for H_0 are given by

$$\dot{\mu}_1 = \frac{\partial H_0}{\partial \phi}, \qquad \dot{\mu}_2 = -\frac{\partial H_0}{\partial \phi} + \frac{\partial H_0}{\partial \psi}, \qquad \dot{\mu}_3 = -\frac{\partial H_0}{\partial \psi},$$
$$\dot{\phi} = \frac{\partial H_0}{\partial \mu_2} - \frac{\partial H_0}{\partial \mu_1}, \qquad \dot{\psi} = \frac{\partial H_0}{\partial \mu_3} - \frac{\partial H_0}{\partial \mu_2}. \qquad (4.8)$$

By regrouping Eqs. (4.8), we see that (μ_1, μ_2, ϕ) can be separated; after solving this system, we can substitute back to get the equations for (μ_3, ψ). Since ψ is the cyclic variable for H_0,

$$\dot{\mu}_3 = 0, \qquad \dot{\psi} = I_3^{-1} - \frac{1}{\Delta} (\tilde{I}_1 \mu_2 - \varepsilon ab \cos\phi \mu_1) \quad (= \omega_3 - \omega_2)$$
$$\equiv \Omega(t).$$

Let $x(t) = (\mu_1(t), \mu_2(t), \phi(t))$ be a homoclinic orbit for the (μ_1, μ_2, ϕ)-dynamics in $\mu_1 + \mu_2 + \mu_3 \equiv M$, $\mu_3 \equiv J$ where M, J are given constants.

As in Holmes and Marsden (1983), if we set

$$\psi(t) = \int_0^t \Omega(s)\, ds + \psi_0,$$

we have only to prove that the *Melnikov function*

$$M(\psi_0)=\int_{-\infty}^{\infty}\left\{H_0,\frac{H_1}{\Omega}\right\}\left(x(t),\int_0^t \Omega(s)\,ds+\psi_0, J\right)dt \tag{4.10}$$

has simple zeroes to get "horseshoes," where $\{\ ,\ \}$ is the bracket in the variables (μ_1, μ_2, ϕ) in $\{\mu_1+\mu_2+\mu_3=M,\ \mu_3=J\}$.

Note that Ω is an explicit function depending on μ_1, μ_2, and so will not be a constant as time changes. For reasons that will become clear, we will consider, instead of the function $M(\psi_0)$, the function

$$N(\psi_0, J):=M(\psi_0)\cdot\frac{1}{\gamma J/I_2};$$

i.e.,

$$-\int_{-\infty}^{\infty}\left\{H_0,\frac{H_1}{\Omega\gamma J/I_3}\right\}\left(x(t),\int_0^t \Omega(s)\,ds+\psi_0, J\right)dt. \tag{4.11}$$

Now,

$$\begin{aligned}H_1'\equiv\frac{H_1}{\gamma J/I_3}= -\frac{1}{\Delta}\,&\{a(\tilde{I}_2\mu_1-\varepsilon ab\mu_2\cos\phi)\cos(\phi+\psi)\\ &+b\cos\psi(\tilde{I}_1\mu_2-\mu_1 ab\varepsilon\cos\phi)\}.\end{aligned}$$

From now on, we will drop $\tilde{}$ on $\tilde{I}_1$ and $\tilde{I}_2$, remembering $\tilde{I}_1\geqslant\varepsilon a^2$ and $\tilde{I}_2\geqslant\varepsilon b^2$. Let us first consider $N(\psi_0, 0)$. Then $\Omega(t)=-(I_1\mu_2-\varepsilon ab\cos\phi\mu_1)/\Delta$, and

$$\begin{aligned}\frac{H_1'}{\Omega}&=\left\{b\cos\psi+a\cos(\phi+\psi)\,\frac{I_2\mu_1-\varepsilon ab\mu_2\cos\phi}{I_1\mu_2-\varepsilon ab\mu_1\cos\phi}\right\}\\ &=\left\{b\left(\cos\int_0^t\Omega(s)\,ds\cdot\cos\psi_0-\sin\int_0^t\Omega(s)\,ds\cdot\sin\psi_0\right)\right.\\ &\quad+a\,\frac{I_2\mu_1-\varepsilon ab\mu_1\cos\psi}{I_2\mu_2-\varepsilon ab\mu_1\cos\psi}\cos\left(\phi+\int_0^t\Omega(s)\,ds\right)\cos\psi_0\\ &\quad\left.-\sin\left(\phi+\int_0^t\Omega(s)\,ds\right)\sin\psi\right\}.\end{aligned} \tag{4.12}$$

We can assume without loss of generality that μ_1 and μ_2 are even functions and ϕ is an odd function; then Ω is an even function and so

$$\int_0^t\Omega(s)\,ds$$

is an odd function of t. Thus

$$\begin{aligned}
\left\{H_0, \frac{H_1'}{\Omega}\right\} &= \left\{H_0, b\cos\int_0^t \Omega(s)\,ds + a\,\frac{I_2\mu_1 - \varepsilon ab\mu_2\cos\phi}{I_1\mu_2 - \varepsilon ab\mu_1\cos\phi}\right. \\
&\quad \left.\times\cos\left(\phi + \int_0^t \Omega(s)\,ds\right)\right\}\cos\psi_0 \\
&\quad - \left\{H_0, b\sin\int_0^t \Omega(s)\,ds + a\,\frac{I_2\mu_1 - \varepsilon ab\mu_2\cos\phi}{I_1\mu_2 - \varepsilon ab\mu_1\cos\phi}\right. \\
&\quad \left.\times\sin\left(\phi + \int_0^t \Omega(s)\,ds\right)\right\}\sin\psi_0 \\
&= \left\{H_0, a\,\frac{I_2\mu_1 - \varepsilon ab\mu_2\cos\phi}{I_1\mu_2 - \varepsilon ab\mu_1\cos\phi}\cos\left(\phi + \int_0^t \Omega(s)\,ds\right)\right\}\cos\psi_0 \\
&\quad - \left\{H_0, a\,\frac{I_2\mu_1 - \varepsilon ab\mu_2\cos\phi}{I_1\mu_2 - \varepsilon ab\mu_1\cos\phi}\sin\left(\phi + \int_0^t \Omega(s)\,ds\right)\right\}\sin\psi_0.
\end{aligned} \tag{4.13}$$

By symmetry, the first term of (4.13) will vanish after integration. Thus,

$$N(\psi_0, 0) = -\int_{-\infty}^{\infty}\left\{H_0, a\,\frac{I_2\mu_1 - \varepsilon ab\mu_2\cos\phi}{I_1\mu_2 - \varepsilon ab\mu_1\cos\phi}\sin\left(\phi + \int_0^t \Omega(s)\,ds\right)\right\}dt\,\sin\phi_0. \tag{4.14}$$

Thus, what we have to do is prove that

$$\int_{-\infty}^{\infty}\left\{H_0, a\,\frac{I_2\mu_1 - \varepsilon ab\mu_2\cos\phi}{I_1\mu_2 - \varepsilon ab\mu_1\cos\phi}\sin\left(\phi + \int_0^t \Omega(s)\,ds\right)\right\}dt \neq 0.$$

Assume, for example, that $I_1 = I_2 = I$ and $a = b = 1$. Then the integrand becomes

$$\begin{aligned}
&\left\{H_0, \frac{I\mu_1 - \varepsilon\mu_2\cos\phi}{I\mu_2 - \varepsilon\mu_1\cos\phi}\sin\left(\phi + \int_0^t \Omega(s)\,ds\right)\right\} \\
&\quad = \dot\phi\,\frac{\partial}{\partial\phi}\left\{\frac{I\mu_1 - \varepsilon\mu_2\cos\phi}{I\mu_2 - \varepsilon\mu_1\cos\phi}\sin\left(\phi + \int_0^t \Omega(s)\,ds\right)\right\} \\
&\quad\quad - \dot\mu_1\left(\frac{\partial}{\partial\mu_2} - \frac{\partial}{\partial\mu_1}\right)\left\{\frac{I\mu_1 - \varepsilon\mu_2\cos\phi}{I\mu_2 - \varepsilon\mu_1\cos\phi}\sin\left(\phi + \int_0^t \Omega(s)\,ds\right)\right\}.
\end{aligned} \tag{4.15}$$

Here

$$\frac{\partial}{\partial\phi}\{\ \}=\frac{\varepsilon I\sin\phi(\mu_2^2-\mu_1^2)}{(I\mu_2-\varepsilon\mu_1\cos\phi)^2}\sin\left(\phi+\int_0^t\Omega(s)\,ds\right)$$
$$+\frac{I\mu_1-\varepsilon\mu_2\cos\phi}{I\mu_2-\varepsilon\mu_1\cos\phi}\cos\left(\phi+\int_0^t\Omega(s)\,ds\right) \tag{4.16}$$

and

$$\left(\frac{\partial}{\partial\mu_2}-\frac{\partial}{\partial\mu_1}\right)\{\ \}=-\frac{(I^2-\varepsilon^2\cos^2\phi)(\mu_1+\mu_2)}{(I\mu_2-\varepsilon\mu_1\cos\phi)^2}\sin\left(\phi+\int_0^t\Omega(s)\,ds\right). \tag{4.17}$$

Let us express μ_1, μ_2 as functions of ϕ for one of the homoclinic orbits that is in the (μ_1,μ_2,ϕ)-dynamics. It is given as the intersection of $\mu_1+\mu_2=M-J$ and

$$H_0=\frac{1}{2\Delta}(I\mu_1^2+I\mu_2^2-2\varepsilon\cos\phi\mu_1\mu_2)=\frac{1}{4}\frac{(M-J)^2}{I-\varepsilon} \tag{4.18}$$

(the energy of the homoclinic orbit).

Set $\nu=\mu_2-\mu_1$. Then we can write

$$H_0=\frac{1}{4}\left(\frac{(M-J)^2}{I+\varepsilon\cos\phi}+\frac{\nu^2}{I-\varepsilon\cos\phi}\right). \tag{4.19}$$

When

$$H_0=\frac{1}{4}\left(\frac{(M-J)^2}{I-\varepsilon}\right),$$

we get

$$\nu=\pm\sqrt{\Gamma}\,(M-J)$$

where

$$\Gamma=\frac{\beta}{1-\beta}(1+\cos\phi)\frac{1-\beta\cos\phi}{1+\beta\cos\phi}\qquad\text{and}\qquad\beta=\frac{\varepsilon}{I}. \tag{4.20}$$

Consider the homoclinic orbit obtained from $\nu=\sqrt{\Gamma}\,(M-J)$. Then,

$$\mu_2=\frac{1+\sqrt{\Gamma}}{2}(M-J)\qquad\text{and}\qquad\mu_1=\frac{1-\sqrt{\Gamma}}{2}(M-J). \tag{4.21}$$

Substitute (4.21) into (4.16) and (4.17) to get

$$\begin{aligned}\frac{\partial}{\partial\phi}\{\ ,\ \} = & \frac{4\beta\sin\phi\sqrt{\Gamma}}{\{(1-\beta\cos\phi)+\sqrt{\Gamma}(1+\beta\cos\phi)\}^2}\\ & \times \sin\left(\phi+\int_0^t \Omega(s)\,ds\right)\\ & + \frac{(1-\beta\cos\phi)-\sqrt{\Gamma}(1+\beta\cos\phi)}{(1-\beta\cos\phi)+\sqrt{\Gamma}(1+\beta\cos\phi)}\\ & \times \cos\left(\phi+\int_0^t \Omega(s)\,ds\right), \end{aligned} \tag{4.22}$$

$$\begin{aligned}\left(\frac{\partial}{\partial\mu_2}-\frac{\partial}{\partial\mu_1}\right)\{\ ,\ \} = & -\frac{4(1-\beta^2\cos^2\phi)(M-J)^{-1}}{\{(1-\beta\cos\phi)+\sqrt{\Gamma}(1+\beta\cos\phi)\}^2}\\ & \times \sin\left(\phi+\int_0^t \Omega(s)\,ds\right). \end{aligned} \tag{4.23}$$

And

$$\dot{\mu}_1 = \frac{d\mu_1}{d\phi}\cdot\dot{\phi} = -\frac{1}{4}(M-J)\cdot\frac{\Gamma'}{\sqrt{\Gamma}}\dot{\phi}.$$

Thus, $N(\psi_0, 0) = A_1 + A_2 + A_3$, where

$$\begin{aligned} A_1 = & \int_{-\infty}^{\infty} \frac{-4\beta\sin\phi\sqrt{\Gamma}\,\dot{\phi}}{((1-\beta\cos\phi)+\sqrt{\Gamma}(1+\beta\cos\phi))^2}\\ & \times \sin\left(\phi+\int_0^t \Omega(s)\,ds\right)dt, \end{aligned} \tag{4.24a}$$

$$\begin{aligned} A_2 = & -\int_{-\infty}^{\infty} \frac{(1-\beta\cos\phi)-\sqrt{\Gamma}(1+\beta\cos\phi)}{(1-\beta\cos\phi)+\sqrt{\Gamma}(1+\beta\cos\phi)}\\ & \times \dot{\phi}\cos\left(\phi+\int_0^t \Omega(s)\,ds\right)dt, \end{aligned} \tag{4.24b}$$

$$\begin{aligned} A_3 = & \int_{-\infty}^{\infty} \frac{1-\beta^2\cos^2\phi}{((1-\beta\cos\phi)+\sqrt{\Gamma}(1+\beta\cos\phi))^2}\frac{\Gamma'}{\sqrt{\Gamma}}\\ & \times \dot{\phi}\sin\left(\phi+\int_0^t \Omega(s)\,ds\right)dt. \end{aligned} \tag{4.24c}$$

Now, note that

$$\Omega = -\frac{1}{\Delta}(I\mu_2 - \varepsilon \cos\phi \mu_1) = -\frac{I\mu_2 - \varepsilon \cos\phi\mu_1}{I^2 - \varepsilon^2 \cos^2\phi}$$

$$= -\frac{1}{I} \cdot \frac{\mu_2 - \beta \cos\phi\mu_1}{1 - \beta^2 \cos^2\phi} \tag{4.25}$$

and that all of the integrand decays exponentially as $t \to \pm\infty$. Therefore, all of the above integrals are analytic functions with respect to $1/I$ and β, $0 < \beta < 1$, $0 < 1/I$, and are continuous in the range $0 < \beta < 1$, $0 \leqslant 1/I$. We will consider the limiting case when $I = \infty$. Then, $\Omega \equiv 0$ and so (4.24) becomes

$$A_1 = \int_{-\infty}^{\infty} \frac{-4\beta \sin \sqrt{\Gamma}\,\dot{\phi}}{((1-\beta\cos\phi) + \sqrt{\Gamma}(1+\beta\cos\phi))^2} \sin\phi\, dt, \tag{4.26a}$$

$$A_2 = -\int_{-\infty}^{\infty} \frac{(1-\beta\cos\phi) - \sqrt{\Gamma}(1+\beta\cos\phi)}{(1-\beta\cos\phi) + \sqrt{\Gamma}(1+\beta\cos\phi)} \dot{\phi} \cos\phi\, dt, \tag{4.26b}$$

$$A_3 = \int_{-\infty}^{\infty} \frac{1-\beta^2\cos^2\phi}{((1-\beta\cos\phi) + \sqrt{\Gamma}(1+\beta\cos\phi))^2} \frac{\Gamma'}{\sqrt{\Gamma}} \dot{\phi} \sin\phi\, dt. \tag{4.26c}$$

Changing variables, these become

$$A_1 = \int_{-\pi}^{\pi} \frac{-4\beta \sin^2\phi \sqrt{\Gamma}\, d\phi}{((1-\beta\cos\phi) + \sqrt{\Gamma}(1+\beta\cos\phi))^2}, \tag{4.27a}$$

$$A_2 = -\int_{-\pi}^{\pi} \frac{(1-\beta\cos\phi) - \sqrt{\Gamma}(1+\beta\cos\phi)}{(1-\beta\cos\phi) + \sqrt{\Gamma}(1+\beta\cos\phi)} \cos\phi\, d\phi, \tag{4.27b}$$

$$A_3 = \int_{-\pi}^{\pi} \frac{1-\beta^2\cos^2\phi}{((1-\beta\cos\phi) + \sqrt{\Gamma}(1+\beta\cos\phi))^2} \frac{\Gamma'}{\sqrt{\Gamma}} \sin\phi\, d\phi. \tag{4.27c}$$

It appears to be difficult to check directly whether the sum $A_1 + A_2 + A_3$ is not equal to zero. To deal with this, let us compare the order of A_1, A_2, and A_3 as $\sqrt{\beta} \to 0$. The order of A_1 is $O(\beta^{3/2})$. For A_2 and A_3, set $\sqrt{\beta} = \delta$ to get

$$A_2 = -\int_{-\pi}^{\pi} \frac{(1-\delta^2\cos\phi) - \dfrac{\delta}{\sqrt{1-\delta^2}}\sqrt{(1+\cos\phi)(1-\delta^4\cos^2\phi)}}{(1-\delta^2\cos\phi) + \dfrac{\delta}{\sqrt{1-\delta^2}}\sqrt{(1+\cos\phi)(1-\delta^4\cos^2\phi)}}$$

$$\times \cos\phi \, d\phi. \tag{4.28}$$

$$= -\int_{-\pi}^{\pi} \frac{1 - \dfrac{\delta}{\sqrt{1-\delta^2}}\sqrt{\dfrac{(1+\cos\phi)(1+\delta^2\cos\phi)}{1-\delta^2\cos\phi}}}{1 + \dfrac{\delta}{\sqrt{1-\delta^2}}\sqrt{\dfrac{(1+\cos\phi)(1+\delta^2\cos\phi)}{1-\delta^2\cos\phi}}} \cos\phi \, d\phi. \tag{4.29}$$

The integrand is equal to

$$\frac{1-\delta(1+\frac{1}{2}\delta^2+O(\delta^4))(1+\delta^2\cos\phi+O(\delta^4))\sqrt{1+\cos\phi}}{1+\delta(1+\frac{1}{2}\delta^2+O(\delta^4))(1+\delta^2\cos\phi+O(\delta^4))\sqrt{1+\cos\phi}}\cos\phi$$

$$\begin{aligned}
&= (1-(\delta+\tfrac{1}{2}\delta^3(1+2\cos\phi)+O(\delta^5))\sqrt{1+\cos\phi}) \\
&\quad \times \{(1-\delta(1+\tfrac{1}{2}\delta^2(1+2\cos\phi)+O(\delta^4))\sqrt{1+\cos\phi}) \\
&\quad + \tfrac{1}{2}(1+\cos\phi)\,\delta^2(1+\tfrac{1}{2}\delta^2(1+2\cos\phi)+O(\delta^4))^2 \\
&\quad + \tfrac{1}{6}(1+\cos\phi)^{3/2}\,\delta^3(1+\tfrac{1}{2}\delta^2(1+2\cos\phi)+O(\delta^4))^3\}\cos\phi + O(\delta^4) \\
&= \cos\phi\{1-2\sqrt{1+\cos\phi}\,\delta+\tfrac{3}{2}(1+\cos\phi)\,\delta^2\}+O(\delta^3).
\end{aligned} \tag{4.30}$$

Now check the coefficients of δ and δ^2 in (4.29); the coefficient of δ is equal to

$$2\int_{-\pi}^{\pi}\sqrt{1+\cos\phi}\cos\phi\,d\phi = \frac{8}{3}\sqrt{2}.$$

For A_3,

$$\Gamma' = \frac{\beta}{1-\beta}\cdot\frac{-\sin\phi(1-\beta^2\cos^2\phi)+2\beta\sin\phi\cos\phi(1+\cos\phi)}{(1+\beta\cos\phi)^2}$$

Thus, the coefficient of $\delta=\sqrt{\beta}$ in the expansion of A_3 is computed as

$$\int_{-\pi}^{\pi}\frac{-\sin^2\phi}{\sqrt{1+\cos\phi}}\,d\phi = -\frac{8}{3}\sqrt{2}.$$

Unfortunately, the first order term in δ in (4.29) vanishes, so we must check the coefficient of δ^2. From (4.30), the coefficient of δ^2 in A_2 equals

$$-\frac{3}{2}\int_{-\pi}^{\pi} \cos\phi(1+\cos\phi)\, d\phi = -6\pi.$$

Similarly, the coefficient of δ^2 in the expansion of A_3 is computed to be

$$\int_{-\pi}^{\pi} 2\sin^2\phi\, d\phi = 4\pi.$$

Thus, the coefficient of δ^2 in $A_1 + A_2 + A_3$ is given by $-6\pi + 4\pi = -2\pi \neq 0$. Therefore, the coefficient of δ^2 in the expansion of $N(\psi_0, 0)$ with respect to β when $I \to \infty$ is not equal to zero. From this, we can conclude that the Melnikov integral has only simple zeros for generic parameter values if the distance between the hinge points and the center of mass of the third body is small. We summarize as follows.

Theorem 4.1. *If the distance between the center of mass of the third body and its hinge point is sufficiently small, then, apart from isolated values of the system parameters, the dynamics of the three body system has Poincaré–Birkhoff–Smale horseshoes, and so is nonintegrable.*

Nonintegrability here means that there are no analytic integrals other than the energy and total linear and angular momentum. The latter is a well-known consequence of the existence of horseshoes [see, for instance, Moser (1973)].

5. DISCUSSION

In this paper, we have developed a fairly complete picture of the dynamics of the planar 3 body system. The kind of analysis that we have presented enables one to study equilibria and their stability, bifurcations of equilibria, periodic and chaotic solutions. Computer graphics of the dynamics illustrating these features has been developed by Sreenath (1987).

While this analysis may be difficult to extend to a complex structure of n bodies, the detailed understanding of the dynamics of 3 bodies helps us to understand the relation among chaos, coherence, and stability in more complex structures and in the continuum limit $n \to \infty$. In fact, it has been proved by Y.-G. Oh that the straight-out configuration of the finite coupled rigid n-body system is always stable, but its continuum analogue turns out not to be formally stable. Moreover, we also have a good understanding of the special structure of periodic orbits bifurcated from the equilibrium for the symmetric coupled rigid body system.

We also point out that the detailed understanding of the Hamiltonian structure via symmetry reduction should assist in the development of numerical algorithms and the control theory for these systems.

In Grossman *et al.* (1988), the dynamics of coupled three-dimensional rigid bodies is studied. The analysis there indicates that there may be a symmetric Hamiltonian Hopf bifurcation leading to interesting periodic and chaotic motions. Again one can conjecture the possibility of interesting three-dimensional waves, such as helical waves, being built from an understanding of the few degrees of freedom situation. An eventual goal is to link this theory with the infinite-dimensional case in Krishnaprasad and Marsden (1988), Simo *et al.* (1988), and Krishnaprasad *et al.* (1988).

NOTE ADDED IN PROOF

After this work was finished, we learned that an infinite dimensional version of Lemma 2.1 was obtained by M. Grillakis in connection with the stability problem for the nonlinear Schrödinger equation (*Comm. Pure and Appl. Math.* **41**, 747–774 (1988)).

ACKNOWLEDGMENTS

This work was partially supported by DOE contract DE-AT03-85ER12097 and by AFOSR-URI grant AFOSR-87-0073 (Y.-G. O. and J. E. M.); and by the National Science Foundation under grant OIR-85-00108, AFOSR-87-0073, and by the Minta Martin Fund for Aeronautical Research (N. S. and P. S. K.).

REFERENCES

Cartan, E. (1928). Sur la stabilité ordinaire des ellipsoides de Jacobi, *Proc. Int. Cong. Math. 1924 (II)*.

Golubitsky, M., and Stewart, I. (1987). Generic bifurcation of Hamiltonian systems with symmetry. *Physica D* **24**, 391–405.

Grossman, R., Krishnaprasad, P. S., and Marsden, J. E. (1988). The dynamics of two coupled rigid bodies. In F. Salam and M. Levi (eds.), *Dynamical Systems Approaches to Nonlinear Problems in Systems and Circuits*, SIAM, Philadelphia, pp. 373–378.

Guillemin, V., and Sternberg, S. (1984). *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge, UK.

Holmes, P., and Marsden, J. E. (1983). Horsehoes and Arnold diffusion for Hamiltonian systems on Lie groups. *Ind. Univ. Math. J.* **32**, 273–310.

Krishnaprasad, P. S., and Marsden, J. E. (1987) Hamiltonian structures and stability of rigid bodies with flexible attachments. *Arch. Rat. Mech. Anal.* **98**, 71–93.

Krishnaprasad, P. S., Marsden, J. E., Posburgh, T., and Simo, J. C. (1988). *Nonlinear Stability of a Rigid Body with an Attached Geometrically Exact Rod.* (In press.)

Montaldi, J., Roberts, M., and Stewart, I. (1988). Periodic solutions near equilibria of symmetric Hamiltonian systems. *Philos. Trans. R. Soc. Lond. Ser. A* **325**, 237–293.

Moser, J. (1973). *Stable and Random Motions in Dynamical Systems*, Princeton University Press, Princeton, NJ.

Moser, J. (1976). Periodic orbits near equilibrium and a theorem by Alan Weinstein. *Commun. Pure Appl. Math.* **29**, 727–747.

Oh, Y.-G. (1987). A stability criterion for Hamiltonian systems with symmetry. *J. Geom. Phys.* **4**, 163–182.

Simo, J. C., Marsden, J. E., and Krishnaprasad, P. S. (1988). The Hamiltonian structure of elasticity: The material, spatial, and connective representation of solids, rods, and plates. *Arch. Rat. Mech. An.* **104**, 125–183.

Sreenath, N. (1987). Modelling and control of complex multibody spacecraft. PhD thesis, University of Maryland, College Park, MD.

Sreenath, N., Oh, Y.-G., Krishnaprasad, P. S., and Marsden, J. E. (1988). The dynamics of coupled planar rigid bodies. I. Reduction, equilibria, and stability. *Dyn. Stability Syst.* **2**, 1–25.

Weinstein, A. (1973). Normal modes for nonlinear Hamiltonian systems. *Inv. Math.* **20**, 47–57.

Weinstein, A. (1978). Bifurcations and Hamilton's principle. *Math. Z.* **159**, 235–248.